LIPSCHITZ GEOMETRY OF COMPLEX SURFACES: ANALYTIC INVARIANTS AND EQUISINGULARITY.

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ABSTRACT. We prove that the outer Lipschitz geometry of a germ $(X, 0)$ of a normal complex surface singularity determines a large amount of its analytic structure. In particular, it follows that any analytic family of normal surface singularities with constant Lipschitz geometry is Zariski equisingular. We also prove a strong converse for families of normal complex hypersurface singularities in $\mathbb{C}^3$: Zariski equisingularity implies Lipschitz triviality. So for such a family Lipschitz triviality, constant Lipschitz geometry and Zariski equisingularity are equivalent to each other.

1. INTRODUCTION

This paper has two aims. The first is to prove the equivalence of Zariski and bilipschitz equisingularity for families of normal complex surface singularities. The second, on which the first partly depends, is to describe which analytic invariants are determined by the Lipschitz geometry of a normal complex surface singularity.

Equisingularity. The question of defining a good notion of equisingularity of a reduced hypersurface $\mathfrak{X} \subset \mathbb{C}^n$ along a non singular complex subspace $Y \subset \mathfrak{X}$ in a neighbourhood of a point $0 \in \mathfrak{X}$ started in 1965 with two papers of Zariski ([26, 27]). This problem has been extensively studied with different approaches and by many authors such as Zariski himself, Abhyankar, Briançon, Gaffney, Hironaka, Lê, Lejeune-Jalabert, Lipman, Mostowski, Parusinski, Pham, Speder, Teissier, Thom, Trotman, Varchenko, Wahl, Whitney and many others.

One of the central concepts introduced by Zariski is the algebro-geometric equisingularity, called nowadays Zariski equisingularity. The idea is that the equisingularity of $\mathfrak{X}$ along Y is defined inductively on the codimension of Y in $\mathfrak{X}$ by requiring that the reduced discriminant locus of a suitably general projection $p\colon \mathfrak{X} \to \mathbb{C}^{n-1}$ be itself equisingular along $p(Y)$. The induction starts by requiring that in codimension one the discriminant locus is nonsingular.

When Y has codimension one in $\mathfrak{X}$, it is well known that Zariski equisingularity is equivalent to the main notions of equisingularity such as Whitney conditions for the pair $(\mathfrak{X} \setminus Y, Y)$ and topological triviality. However these properties fail to be equivalent in higher codimension: Zariski equisingularity still implies topological triviality ([23, 24]) and Whitney conditions ([17]), but the converse statements are false ([4, 5, 22]) and a global theory of equisingularity is still far from being established. For good surveys of equisingularity questions, see [13, 20].

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The first main result of this paper states the equivalence between Zariski equisingularity, constancy of Lipschitz geometry and triviality of Lipschitz geometry in the case Y is the singular locus of $\mathfrak{X}$ and has codimension 2 in $\mathfrak{X}$. We must say what we mean by "Lipschitz geometry". If $(X,0)$ is a germ of a complex variety, then any embedding $\phi\colon (X,0) \hookrightarrow (\mathbb{C}^n,0)$ determines two metrics on $(X,0)$: the outer metric

$$d_{out}(x_1,x_2) := ||\phi(x_1)-\phi(x_2)|| \quad (i.e., \text{ distance in } \mathbb{C}^n)$$

and the inner metric

$$d_{inn}(x_1,x_2) := \inf\{\text{length}(\phi\circ\gamma) : \gamma \text{ is a rectifyable path in } X \text{ from } x_1 \text{ to } x_2\}.$$

The outer metric determines the inner metric, and up to bilipschitz equivalence both these metrics are independent of the choice of complex embedding. We speak of the *Lipschitz geometry* of $(X,0)$. If we work up to semi-algebraic bilipschitz equivalence, we speak of the *semi-algebraic Lipschitz geometry.*

We consider here the case of normal complex surface singularities. The inner metric in this case has been well studied, and is completely classified in [1]. The present paper is entirely devoted to outer geometry.

Let $(\mathfrak{X},0)\subset(\mathbb{C}^n,0)$ be a germ of hypersurface at the origin of $\mathbb{C}^n$ with smooth codimension 2 singular set $(Y,0)\subset(\mathfrak{X},0)$.

The germ $(\mathfrak{X},0)$ *has constant (semi-algebraic) Lipschitz geometry* along Y if there exists a smooth (semi-algebraic) retraction $r\colon(\mathfrak{X},0)\to(Y,0)$ whose fibers are transverse to Y there is a neighbourhood U of 0 in Y such that for all $y\in U$ there exists a (semi-algebraic) bilipschitz diffeomorphism $h_y\colon (r^{-1}(y),y)\to(r^{-1}(0)\cap\mathfrak{X},0)$.

The germ $(\mathfrak{X},0)$ is (semi-algebraic) *Lipschitz trivial* along Y if there exists a germ at 0 of a (semi-algebraic) bilipschitz homeomorphism $\Phi\colon(\mathfrak{X},Y)\to(X,0)\times Y)$ with $\Phi|_Y = id_Y$, where $(X,0)$ is a normal complex surface germ.

Theorem 1.1. *The following are equivalent:*

1. $(\mathfrak{X},0)$ *is Zariski equisingular along* Y*;*
2. $(\mathfrak{X},0)$ *has constant semi-algebraic Lipschitz geometry along* Y*;*
3. $(\mathfrak{X},0)$ *is semi-algebraic Lipschitz trivial along* Y

The equivalence between (1) and (3) has been conjectured by T. Mostowski in a talk for which written notes are available ([14]). He also gave insights to prove the result using his theory of Lipschitz stratifications. Our approach is different and we construct a decomposition of the pair $(\mathfrak{X},\mathfrak{X}\setminus Y)$ using the theory of carrousels, introduced by D. T. Lê in [10], on the family of discriminant curves.

The implication (3)⇒(2) is trivial. The implication (2)⇒(1) will be an easy consequence of item (2) of Theorem 1.3 below. The final part of the paper is devoted to the proof of (1)⇒(3).

Notice that it was known already that the inner Lipschitz geometry is not sufficient to understand Zariski equisingularity. Indeed, the family of hypersurfaces $(X_t,0)\subset(\mathbb{C}^3,0)$ with equation $z^3+tx^4z+x^6+y^6=0$ is not Zariski equisingular (see [6]); at $t=0$, the discriminant curve has 6 branches, while it has 12 branches when $t\neq0$), while it follows from [1] that it has constant inner geometry (in fact $(X_t,0)$ is metrically conical for each t close to 0).

Invariants from Lipschitz geometry. The other main results of this paper are described in the next two theorems, which deal respectively with Lipschitz and semi-algebraic Lipschitz geometry:

Theorem 1.2. *If $(X,0)$ is a normal complex surface singularity then the outer Lipschitz geometry on X determines:*

(1) *the decorated resolution graph of the minimal good resolution of $(X,0)$ which resolves the basepoints of a general linear system of hyperplane sections;*
(2) *the multiplicity of $(X,0)$ and the maximal ideal cycle in its resolution;*
(3) *for a general hyperplane H, the outer geometry of the curve $(X\cap H,0)$.*

By "decorated resolution graph" we mean the resolution graph decorated with arrows corresponding to the strict transforms of the resolved curve.

Theorem 1.3. *If $(X,0)$ is a normal complex surface singularity then its semi-algebraic outer geometry determines:*

(1) *the decorated resolution graph of the minimal good resolution of $(X,0)$ which resolves the basepoints of the family of polar curves of plane projections;*
(2) *the topology of the discriminant curve of a general plane projection;*
(3) *the outer geometry of the polar curve $(\Pi,0)$ of a general plane projection.*

Item (3) of this theorem is a straightforward consequence of item (2). The argument is as follows: by Pham-Teissier [16] (see also Fernandes [8]), the outer geometry of a complex curve in $\mathbb{C}^n$ is determined by the topology of a general plane projection of this curve. By Teissier [21, page 462 Lemme 1.2.2 ii)], if one takes a general plane projection $\ell\colon (X,0)\to(\mathbb{C}^2,0)$, then this projection is general for its polar curve Π. Therefore the topology of the discriminant curve determines the outer geometry of the polar curve.

This paper has four parts, of which the final two are devoted to proving the above theorems: Theorems 1.2 and 1.3 in Part 3 (Sections 7 to 9) and the completion of the proof of 1.1 in Part 4 (Sections 10 to 17). Parts 1 and 2 are preparatory, introducing concepts and techniques that are needed later.

In particular, Part 1 proves a stronger version of the result of Pham and Teissier [16] (see also Fernandes [8]) that the outer Lipschitz geometry of a plane curve germ determines its embedded topology. Our approach only needs the geometry up to bilipschitz equivalence, with no analytic or smoothness requirements.

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Part 1: Carrousel and geometry of curves

2. The carrousel of a plane curve germ

Let $(C,0)\subset(\mathbb{C}^2,0)$ be a reduced curve germ. We will describe a *carrousel decomposition* of the germ $(\mathbb{C}^2,0)$ with respect to the curve C. We use a much finer

decomposition than is needed to just understand the geometry of a plane curve, since we need to build on it in future sections of the paper. The more usual version of the carrousel for a plane curve will be described at the end of this section, by amalgamating certain pieces of our carrousel.

The tangent space to C at 0 is a union $\bigcup_{j=1}^m L^{(j)}$ of lines. For each j we denote the union of components of C which are tangent to $L^{(j)}$ by $C^{(j)}$. We can assume our coordinates (x,y) in $\mathbb{C}^2$ are chosen so that the y-axis is transverse to each $L^{(j)}$.

We choose $\epsilon_0 > 0$ sufficiently small that the set $\{(x,y) : |x| = \epsilon\}$ is transverse to C for all $\epsilon \le \epsilon_0$. We define conical sets $V^{(j)}$ of the form

$$V^{(j)} := \{(x,y) : |y - a_1^{(j)}x| \le \eta|x|, |x| \le \epsilon_0\} \subset \mathbb{C}^2\,,$$

where the equation of the line $L^{(j)}$ is $y = a_1^{(j)}x$ and $\eta > 0$ is small enough that the cones are disjoint except at 0. If ϵ_0 is small enough $C^{(j)} \cap \{|x| \le \epsilon_0\}$ will lie completely in $V^{(j)}$.

There is then an $R > 0$ such that for any $\epsilon \le \epsilon_0$ the sets $V^{(j)}$ meet the boundary of the "square ball"

$$B_\epsilon := \{(x, y \in \mathbb{C}^2 : |x| \le \epsilon, |y| \le R\epsilon\}$$

only in the part $|x| = \epsilon$ of the boundary. We will use these balls as a system of Milnor balls.

We now describe the carrousel decomposition of $V^{(j)}$ (see [1]). We will fix j for the moment and therefore drop the superscripts, so our tangent line L has equation $y = a_1x$. The collection of coefficients and exponents appearing in the following description depends, of course, on $j = 1, \dots, m$ (except that p_1 is 1 for all but at most one j).

We first truncate the Puiseux series for each component of C at a point where truncation does not affect the topology of C. Then for each pair $\kappa = (f, p_k)$ consisting of a Puiseux polynomial $f = \sum_{i=1}^{k-1} a_i x^{p_i}$ and an exponent p_k for which there is a Puiseux series $y = \sum_{i=1}^{k} a_i x^{p_i} + \dots$ describing some component of C, we consider all components of C which fit this data. If $a_{k1}, \dots, a_{km_\kappa}$ are the coefficients of x^{p_k} which occur in these Puiseux polynomials we define

$$\begin{aligned} B_\kappa := \Big\{(x,y) : \alpha_\kappa|x^{p_k}| \le |y - \sum_{i=1}^{k-1} a_i x^{p_i}| \le \beta_\kappa |x^{p_k}| \\ |y - (\sum_{i=1}^{k-1} a_i x^{p_i} + a_{kj}x^{p_k})| \ge \gamma_\kappa |x^{p_k}| \text{ for } j = 1, \dots, m_\kappa \Big\}\,. \end{aligned}$$

Here $\alpha_\kappa, \beta_\kappa, \gamma_\kappa$ are chosen so that $\alpha_\kappa < |a_{kj}| - \gamma_\kappa < |a_{kj}| + \gamma_\kappa < \beta_\kappa$ for each $j = 1, \dots, m_\kappa$. If ϵ is small enough, the sets B_κ will be disjoint for different κ.

The intersection $B_\kappa \cap \{x = t\}$ is a finite collection of disks with some smaller disks removed. The diameter of each of them is $O(t^{p_k})$. We call p_k the *rate* of B_κ since it describes a rate of shrink of the components of the slices $B_\kappa \cap \{x = t\}$ as $|t|$ tends to 0, and we call B_κ a $B(p_k)$*-piece*, or simply a B*-piece* if the rate is not needed. Other rates, related to geometric decompositions of a normal complex surface and expressing the same idea of speed of shrinking, appear later in this paper (Sections 5 and 6).

The closure of the complement in $V = V^{(j)}$ of the union of the B_κ's is a union of pieces each of which has link either a solid torus or a "toral annulus" (annulus

$\times\mathbb{S}^1$). We call the latter *A-pieces* and the ones with solid torus link *D-pieces*, or if their rates are relevant, *$A(q,q')$-piece* or *$D(q)$-piece* (an A-piece has two associated rates $q < q'$, namely the rates of its outer and inner boundaries; A-pieces with $q = q'$ will also be used later).

We obtain what we call a carrousel decomposition of $V = V^{(j)}$. More generally:

Definition 2.1. A *carrousel decomposition* of a cone $V \subset B_\epsilon$ is a decomposition of V as a union of B-, A- or D-pieces. A *carrousel section* is the picture of the intersection of a carrousel decomposition with a line $x = t$.

In fact we will use A-, B- and D-pieces later in a more general setting (see also [1, Section 11]).

We call $\overline{B_\epsilon \setminus \bigcup V^{(j)}}$ a $B(1)$ piece, even though it may have A- or D-topology. It is metrically conical, and together with the carrousel decompositions of the $V^{(j)}$'s we get a carrousel decomposition of the whole of B_ϵ.

Example 2.2. Figure 1 shows a carrousel section for C having two branches with Puiseux expansions respectively $y = ax^{4/3} + bx^{13/6} + \dots$ and $y = cx^{7/4} + \dots$. Note that the intersection of a piece of the decomposition of V with the disk $V \cap \{x = \epsilon\}$ will usually have several components. Note also that the rates in A- and D- pieces are determined by the rates in the neighbouring B- pieces.

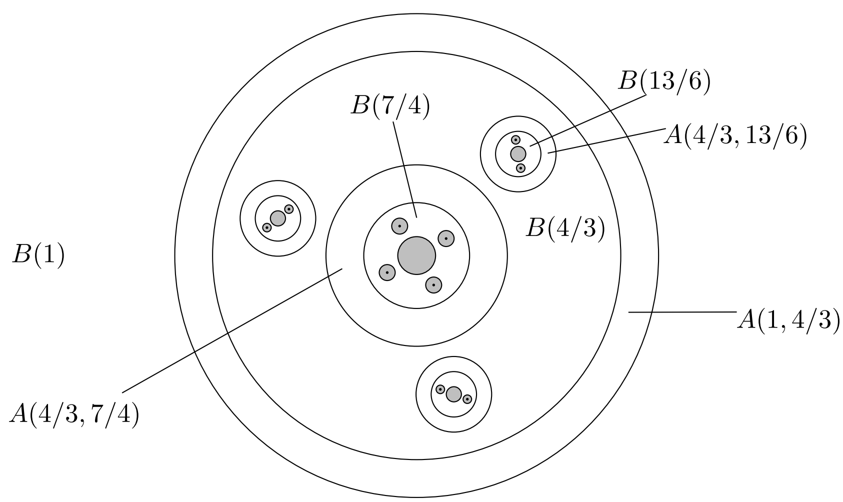


FIGURE 1. Carrousel section for $C = \{y = ax^{4/3} + bx^{13/6} + \dots\} \cup \{y = cx^{7/4} + \dots\}$. The D-pieces are gray.

Plain carrousel decomposition. For the study of plane curves a much simpler carrousel decomposition suffices. We take the above decomposition of $V = V^{(j)}$ and iteratively amalgamate pieces as follows (we use a similar but different amalgamation process in [1, Section 13] and later in this paper). We amalgamate any D-piece that does not contain part of C with the piece that meets it along its boundary, and we amalgamate any A-piece with the piece that meets it along its outer boundary. This may produce new D- or A-pieces and we repeat the amalgamation iteratively until no further amalgamation is possible. We then discard the outermost A-piece

whose outer boundary is $\partial V^{(j)}$, at which point we only have D-pieces, each of which is a disk neighbourhood of a component of C, and B-pieces. The B-pieces still have their rates associated with them (rate of the outer boundary).

We call this simplified carrousel the *plain carrousel*. Figure 2 shows the section of the plain version of the carrousel of Figure 1.

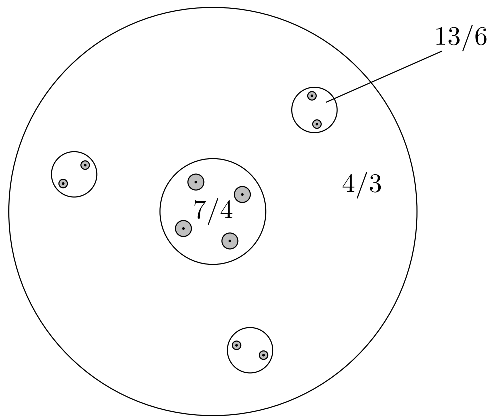


FIGURE 2. Plain carrousel section for $C = \{y = ax^{4/3} + bx^{13/6} + \dots\} \cup \{y = cx^{7/4} + \dots\}$.

The combinatorics of the plain carrousel section can be encoded by a rooted tree, with vertices corresponding to pieces, edges corresponding to pieces which intersect along a circle, and rational weights associated to the nodes of the tree. We call this the *combinatorial carrousel*. It is easy to recover the embedded topology of the plane curve from the combinatorial carrousel. For a careful description of how to do this in terms of either the Eggers tree or the Eisenbud-Neumann splice diagram of the curve see C.T.C. Wall's book [25]. In any case, we record here:

Proposition 2.3. *The combinatorial carrousel for a plane curve germ determines its embedded topology.* $\square$

3. Lipschitz geometry and topology of a plane curve

In this section we prove the following strong version of a result of Pham and Teissier [16] (see also [8]) about plane curve germs. What is new is that we consider the germ just as metric space germ up to bilipschitz equivalence, without the analytic restrictions of [8, 16].

Proposition 3.1. *The outer Lipschitz geometry of a plane curve germ* $(C, 0) \subset (\mathbb{C}^2, 0)$ *determines its embedded topology.*

More generally, if $(C, 0) \subset (\mathbb{C}^n, 0)$ *is a curve germ and* $\ell\colon \mathbb{C}^n \to \mathbb{C}^2$ *is a generic plane projection then the outer Lipschitz geometry of* $(C, 0)$ *determines the embedded topology of the plane projection* $(\ell(C), 0) \subset (\mathbb{C}^2, 0)$.

Remark 3.2. The converse result, that the embedded topology of a plane curve determines the outer Lipschitz geometry, is easier, and is proved in [16].

Proof. The more general statement follows immediately from the plane curve case since Pham and Teissier prove in [16] that for a generic plane projection ℓ the restriction $\ell|_C\colon (C,0) \to (\ell(C),0)$ is bilipschitz for the outer geometry on C. So we assume from now on that $n=2$, so $(C,0)\subset(\mathbb{C}^2,0)$ is a plane curve.

In view of Proposition 2.3 we need to describe how to recover the carrousel section from the outer Lipschitz geometry. We will first describe how to recover it using analytic structure and outer geometry and then describe the adjustment needed to recover it after forgetting the analytic structure and allowing a bilipschitz change of the geometry.

We assume, as in the previous section, that the tangent space to C at 0 is a union of lines $L^{(j)}$, $j=1,\dots,m$, which are all transverse to the y-axis. We let B_ϵ, $\epsilon\le\epsilon_0$ be the family of Milnor balls for C of the previous section, with boundaries $S_\epsilon=\partial B_\epsilon$.

The lines $x=t$ for $t\in(0,\epsilon_0]$ intersect C in a finite set of points $p_1(t),\dots,p_\mu(t)$ where μ is the multiplicity of C. For each $0<j<k\le\mu_i$ the distance $d(p_j(t),p_k(t))$ has the form $O(t^{q_{jk}})$, where q_{jk} is either an essential Puiseux exponent for a branch of the plane curve C or a coincidence exponent between two branches of C.

Lemma 3.3. *The map $\{(j,k)\mid 1\le j<k\le\mu_i\}\mapsto q_{jk}$ determines the embedded topology of C.*

Proof. By Proposition 2.3 it suffices to prove that the map $\{(j,k)\mid 1\le j<k\le \mu_i\}\mapsto q_{jk}$ determines the combinatorial carrousel of the curve C.

Let $q_1>q_2>\ldots>q_s$ be the images of the map $(j,k)\mapsto q_{jk}$. The proof consists of reconstructing a topological version of the carrousel section of C from the innermost pieces to the outermost ones by an inductive process starting with q_1 and ending with q_s. This construction has a certain analogy with that described later in the proof of (1) of Proposition 9.1.

We start with μ discs $D_1^{(0)},\dots,D_\mu^{(0)}$, which will be the innermost pieces of the carrousel. We consider the graph $G^{(1)}$ whose vertices are in bijection with these μ disks and with an edge between vertices (j) and (k) if and only if $q_{jk}=q_1$. Let $G_1^{(1)},\dots,G_{\nu_1}^{(1)}$ be the connected components of $G^{(1)}$ and denote by $v_k^{(1)}$ the number of vertices of $G_k^{(1)}$. For each $G_k^{(1)}$ with $v_k^{(1)}>1$ we consider a disc $B_k^{(1)}$ with $v_k^{(1)}$ holes, and we glue the discs $D_j^{(0)}$, $(j)\in\operatorname{vert}G_k^{(1)}$, into the inner boundary components of $B_k^{(1)}$. For a $G_k^{(1)}$ with just one vertex, (j_k) say, we rename $D_{j_k}^{(0)}$ as $D_k^{(1)}$.

The numbers q_{jk} have the property that $q_{jl}=min(q_{jk},q_{kl})$ for any triple j,k,l. So for each distinct m,n the number $q_{j_mk_n}$ does not depend on the choice of vertices (j_m) in $G_m^{(1)}$ and (k_n) in $G_n^{(1)}$.

We iterate the above process as follows: we consider the graph $G^{(2)}$ whose vertices are in bijection with the connected components $G_1^{(1)},\dots,G_{\nu_1}^{(1)}$ and with an edge between the vertices $(G_m^{(1)})$ and $(G_n^{(1)})$ if and only if $q_{j_mk_n}$ equals q_2 (with vertices (j_m) and (j_n) in $G_m^{(1)}$ and $G_n^{(1)}$ respectively). Let $G_1^{(2)},\dots,G_{\nu_2}^{(2)}$ be the connected components of G_2. For each $G_k^{(2)}$ let $v_k^{(2)}$ be the number of its vertices. If $v_k^{(2)}>1$ we take a disc $B_k^{(2)}$ with $v_k^{(2)}$ holes and glue the corresponding pieces ($B_l^{(1)}$'s or $D_l^{(1)}$'s) into these holes. If $v_k^{(2)}=1$ we rename the corresponding B- or D-piece to $B_k^{(2)}$ respectively $D_k^{(2)}$.

By construction, repeating this process for s steps gives a topological version of the carrousel section of the curve C, and hence its embedded topology. $\square$

As already noted, this discovery of the topology involves the complex structure and outer metric. We must show that we can do it without use of the complex structure, even after applying a bilipschitz change to the outer metric.

Recall that we denote by $C^{(j)}$ the part of C tangent to the line $L^{(j)}$. We first note that it suffices to discover the topology of each $C^{(j)}$ independently, since the $C^{(j)}$'s are distinguished by the fact that the distance between any two of them outside a ball of radius ϵ around 0 is $O(\epsilon)$, and this is true even after bilipschitz change to the metric. We will therefore assume from now on that the tangent to C is a single complex line.

Our points $p_1(t),\dots,p_\mu(t)$ that we used to find the numbers q_{jk} were obtained by intersecting with the line $x=t$. The arc $p_1(t)$, $t\in[0,\epsilon]$ satisfies $d(0,p_1(t))=O(t)$. Moreover, the other points $p_2(t),\dots,p_\mu(t)$ are in the transverse disk of radius kt centered at $p_1(t)$ in the plane $x=t$ (k can be as small as we like, so long as ϵ is then chosen sufficiently small).

Instead of a transverse disk of radius kt, we can use a ball $B(p_1(t),kt)$ of radius kt centered at $p_1(t)$. This $B(p_1(t),kt)$ intersects C in μ disks $D_1(t),\dots,D_\mu(t)$, and we have $d(D_j(t),D_k(t))=O(t^{q_{jk}})$, so we still recover the numbers q_{jk}. In fact, the ball in the outer metric on C of radius kt around $p_1(t)$ is $B_C(p_1(t),kt):=C\cap B(p_1(t),kt)$, which consists of these μ disks $D_1(t),\dots,D_\mu(t)$.

Now we can replace the arc $p_1(t)$ by any continuous arc $p'_1(t)$ on C with the property that $d(0,p'_1(t))=O(t)$, and if k is sufficiently small it is still true that $B_C(p'_1(t),kt)$ consists of μ disks $D'_1(t),\dots,D'_\mu(t)$ with $d(D'_j(t),D'_k(t))=O(t^{q_{jk}})$. So at this point, we have gotten rid of the dependence on analytic structure in discovering the topology, but not yet dependence on the outer geometry.

A K-bilipschitz change to the metric may make the components of $B_C(p'_1(t),kt)$ disintegrate into many pieces, so we can no longer simply use distance between pieces. To resolve this, we consider both $B'_C(p'_1(t),kt)$ and $B'_C(p'_1(t),\frac{k}{K^4}t)$ where B' means we are using the modified metric. Then only μ components of $B'_C(p_1(t),kt)$ will intersect $B'_C(p_1(t),\frac{k}{K^4}t)$. Naming these components $D'_1(t),\dots,D'_\mu(t)$ again, we still have $d(D'_j(t),D'_k(t))=O(t^{q_{jk}})$ so the q_{jk} are determined as before. $\square$

Part 2: Geometric decompositions of a normal complex surface singularity

4. Polar zones

We denote by $\mathbf{G}(k,n)$ the Grassmannian of k-dimensional subspaces of $\mathbb{C}^n$.

Definition 4.1 (General linear projection)**.** Let $(X,0)\subset(\mathbb{C}^n,0)$ be a normal complex surface germ. For $\mathcal{D}\in\mathbf{G}(n-2,n)$ let $\ell_{\mathcal{D}}\colon\mathbb{C}^n\to\mathbb{C}^2$ be the linear projection $\mathbb{C}^n\to\mathbb{C}^2$ with kernel $\mathcal{D}$. Let $\Pi_{\mathcal{D}}$ be the polar of $(X,0)$ for this projection, *i.e.*, the closure in $(X,0)$ of the singular locus of the restriction of $\ell_{\mathcal{D}}$ to $X\smallsetminus\{0\}$, and let $\Delta_{\mathcal{D}}=\ell_{\mathcal{D}}(\Pi_{\mathcal{D}})$ be the discriminant curve. There exists an open dense subset $\Omega\subset\mathbf{G}(n-2,n)$ such that $\{(\Pi_{\mathcal{D}},0):\mathcal{D}\in\Omega\}$ forms an equisingular family of curve germs in terms of strong simultaneous resolution and such that the discriminant curves $\Delta_{\mathcal{D}}$ are reduced and no tangent line to $\Pi_{\mathcal{D}}$ at 0 is contained in $\mathcal{D}$ ([11, (2.2.2)] and [21, V. (1.2.2)]). The projection $\ell_{\mathcal{D}}\colon\mathbb{C}^n\to\mathbb{C}^2$ is *general* for $(X,0)$ if $\mathcal{D}\in\Omega$.

The condition $\Delta_{\mathcal{D}}$ reduced means that any $p \in \Delta_{\mathcal{D}} \setminus \{0\}$ has a neighbourhood U in $\mathbb{C}^2$ such that one component of $(\ell_{\mathcal{D}}|_X)^{-1}(U)$ maps by a two-fold branched cover to U and the other components map bijectively.

Definition 4.2 (Nash Modification)**.** Let $\lambda \colon X \setminus \{0\} \to \mathbf{G}(2,n)$ be the map which maps $x \in X \setminus \{0\}$ to the tangent plane $T_x X$. The closure $\check{X}$ of the graph of λ in $X \times \mathbf{G}(2,n)$ is a reduced analytic surface. The *Nash modification of* $(X,0)$ is the induced morphism $\nu \colon \check{X} \to X$.

According to [18, Part III,Theorem 1.2], a resolution of $(X,0)$ factors through the Nash modification if and only if it has no basepoints for the family of polar curves $\Pi_{\mathcal{D}}$ parametrized by $\mathcal{D} \in \Omega$.

In this section, we consider the minimal good resolution $\pi \colon (\widetilde{X}, E) \to (X,0)$ of $(X,0)$ for which this property holds and we denote by Γ the dual resolution graph. Let $E_1, \dots, E_r$ be the exceptional curves in E. We denote by v_k the vertex of Γ corresponding to E_k.

Notations. For each $k = 1, \dots, r$ let $N(E_k)$ be a small closed tubular neighbourhood of E_k and let

$$\mathcal{N}(E_k) = \overline{N(E_k) \setminus \bigcup_{k' \neq k} N(E_{k'})}.$$

For any subgraph Γ' of Γ define:

$$N(\Gamma') := \bigcup_{v_k \in \Gamma'} N(E_k) \quad \text{and} \quad \mathcal{N}(\Gamma') := \overline{N(\Gamma) \setminus \bigcup_{v_k \notin \Gamma'} N(E_k)}\,.$$

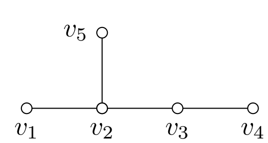
Assume ϵ_0 is sufficiently small that $\pi^{-1}(X \cap \mathbb{B}_{\epsilon_0})$ is included in $N(\Gamma)$.

Definition 4.3. A $\mathcal{P}$*-curve* ($\mathcal{P}$ for "polar") will be an exceptional curve in $\pi^{-1}(0)$ which intersects the strict transform of the polar curve of any general linear projection. A vertex of the resolution graph of π which represents a $\mathcal{P}$-curve is a $\mathcal{P}$*-node*. A $\mathcal{P}$*-Tjurina component* of Γ is any maximal connected subgraph of Γ which does not contain a $\mathcal{P}$-node. If $E_i \subset \pi^{-1}(0)$ is a $\mathcal{P}$-curve, we call $\pi(\mathcal{N}(E_i))$ the *polar zone* associated with E_i.

There is a rational number q_i associated with a polar zone $\pi(\mathcal{N}(E_i))$, defined as the degree of contact of the images in X of any two curvettes on E_i, or equivalently as the degree of contact at 0 of their images by any generic plane projection.

Definition 4.4. We call q_i the *rate* of the polar-zone $\pi(\mathcal{N}(E_i))$, since it is a "rate" in the sense of the rates of carrousel pieces discussed in Section 2.

Example 4.5. Let $(X,0)$ be the D_5 singularity with equation $x^2y + y^4 + z^2 = 0$. Let $v_1, \dots, v_5$ be the vertices of its minimal resolution graph indexed as follows (all Euler weights are -2):



As we shall see, this is not yet the resolution which resolves the family of polars.

The total transform by π of the coordinate functions x, y and z are:

$$(x\circ\pi) = 2E_1 + 3E_2 + 2E_3 + E_4 + 2E_5 + x^*$$
$$(y\circ\pi) = E_1 + 2E_2 + 2E_3 + 2E_4 + E_5 + y^*$$
$$(z\circ\pi) = 2E_1 + 4E_2 + 3E_3 + 2E_4 + 2E_5 + z^*\,,$$

where * means strict transform.

Set $f(x,y,z) = x^2y + y^4 + z^2$. The polar curve Π of a general linear projection $\ell\colon (X,0) \to (\mathbb{C}^2,0)$ has equation $g=0$ where g is a general linear combination of the partial derivatives $f_x = 2xy$, $f_y = x^2 + 4y^3$ and $f_z = 2z$. The multiplicities of g are given by the minimum of the compact part of the three divisors

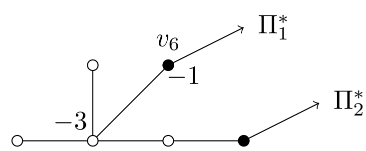
$$(f_x\circ\pi) = 3E_1 + 5E_2 + 4E_3 + 3E_4 + 3E_5 + f_x^*$$
$$(f_y\circ\pi) = 3E_1 + 6E_2 + 4E_3 + 2E_4 + 3E_5 + f_y^*$$
$$(f_z\circ\pi) = 2E_1 + 4E_2 + 3E_3 + 2E_4 + 2E_5 + f_z^*\,.$$

So the total transform of g is equal to:

$$(g\circ\pi) = 2E_1 + 4E_2 + 3E_3 + 2E_4 + 2E_5 + \Pi^*\,.$$

In particular, Π is resolved by π and its strict transform Π^* has two components Π_1 and Π_2, which intersect respectively E_2 and E_4.

Since the multiplicities $m_2(f_x) = 5$, $m_2(f_y) = 4$ and $m_2(z) = 6$ along E_2 are distinct, the family of polar curves of general plane projections has a base point on E_2 and one must blow-up once to resolve the basepoint, creating a new exceptional curve E_6 and a new vertex v_6 in the graph. So we obtain two $\mathcal{P}$-nodes which are represented as black vertices in the resolution graph (omitted weights are -2):



We will compute the rates of the polar zones $\pi(\mathcal{N}(E_4))$ and $\pi(\mathcal{N}(E_6))$ in Example 8.12; they equal 2 and 5/2 respectively.

Let us fix a $\mathcal{D} \in \Omega$ (see Definition 4.1). In the proof of [1, Corollary 3.4] we construct a disk neighbourhood A of the strict transform $\Pi^*_{\mathcal{D}}$ of $\Pi_{\mathcal{D}}$ by π which is the union of a family of disjoint strict transforms of polars $\Pi^*_{\mathcal{D}_t}$ parametrized by t in a disk neighbourhood of 0 in $\mathbb{C}$ and with $\mathcal{D}_0 = \mathcal{D}$.

Definition 4.6. We call the image $\pi(A)$ a *very thin zone* about $\Pi_{\mathcal{D}}$ and its image by $\ell_{\mathcal{D}}$ a *very thin zone* about the discriminant curve $\Delta_{\mathcal{D}}$. If Π' is a component of $\Pi_{\mathcal{D}}$, we call the closure of the connected component of $\pi(A) \smallsetminus \{0\}$ which intersect Π' a *very thin zone* about Π' and its image by $\ell_{\mathcal{D}}$ a *very thin zone* about the branch $\ell_{\mathcal{D}}(\Pi')$ of $\Delta_{\mathcal{D}}$.

For $x \in X$, we define the *local bilipschitz constant* $K(x)$ of the projection $\ell_{\mathcal{D}}\colon X \to \mathbb{C}^2$ as follows: $K(x)$ is infinite if x belongs to the polar curve $\Pi_{\mathcal{D}}$ and at a point $x \in X \smallsetminus \Pi_{\mathcal{D}}$ it is the reciprocal of the shortest length among images of unit vectors in T_xX under the projection $d\ell_{\mathcal{D}}\colon T_xX \to \mathbb{C}^2$.

For $K_0 > 1$ we denote $\mathcal{B}_{K_0}$ the set of points $x \in X$ where $K(x) \geq K_0$. As a consequence of [1, 3.3, 3.4], we obtain that for $K_0 > 0$ sufficiently large, the set $\mathcal{B}_{K_0}$ can be approximated by a very thin zone about $\Pi_{\mathcal{D}}$. Precisely:

Lemma 4.7. *Let VT be a very thin zone about Π. There exists $K_0, K_1 \in \mathbb{R}$ with $1 < K_1 < K_0$ such that $\mathcal{B}_{K_0} \subset VT \subset \mathcal{B}_{K_1}$.* $\square$

Recall that $\pi\colon \widetilde{X} \to X$ factors through the Nash modification $\nu\colon \check{X} \to X$. Let $\sigma\colon \widetilde{X} \to \mathbf{G}(2,n)$ be the map induced by the projection $p_2\colon \check{X} \subset X \times \mathbf{G}(2,n) \to \mathbf{G}(2,n)$. The map σ is well defined on $\pi^{-1}(0)$ and according to [9, Section 2] (see also [18, Part III, Theorem 1.2]), it is constant on any connected component of the complement of $\mathcal{P}$-curves in $\pi^{-1}(0)$. We will need later the following lemma about limits of tangent planes, which follows from this:

Lemma 4.8. (1) *Let Γ' be a $\mathcal{P}$-Tjurina component of Γ (Definition 4.3). There exists $P_{\Gamma'} \in \mathbf{G}(2,n)$ such that $\lim_{t\to 0} T_{\gamma(t)}X = P_{\Gamma'}$ for any real arc $\gamma\colon ([0,1),0) \to (N(\Gamma'),0)$ whose degree of contact with any polar zone with rate q differs from q.*

(2) *Let $E_k \subset \pi^{-1}(0)$ be a $\mathcal{P}$-curve and $x \in E_k$ be a smooth point of the exceptional divisor $\pi^{-1}(0)$. There exists a plane $P_x \in \mathbf{G}(2,n)$ such that $\lim_{t\to 0} T_{\gamma(t)}X = P_x$ for any real arc $\gamma\colon ([0,1),0) \to (\pi(\mathcal{N}(E_k)),0)$ whose strict transform meets $\pi^{-1}(0)$ at x.* $\square$

Figure 3 represents schematically a polar zone (in gray).

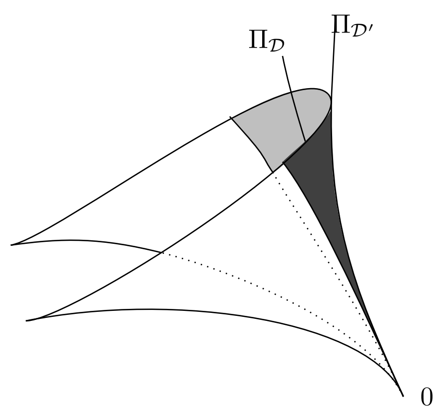


FIGURE 3. Polar zone

In the sequel, we will often represent *real slices* of this picture as in Figure 4, *i.e.*, the intersection $X \cap \{h = t\} \cap P$ where P is a general real $2n-1$-plane in $\mathbb{C}^n \cong \mathbb{R}^{2n}$. The polar zone is there represented by the thick arc. In such a picture, an arc $\gamma\colon [0,1) \to X$ on P such that $||\gamma(t)|| = t$ is a point. In particular, the intersection of the polar curve Π_ℓ with P consists of a finite number of real arcs, and then its intersection with a real slice is a finite number of points.

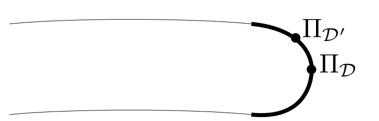


FIGURE 4. Real slice

5. Geometric decomposition of $(X, 0)$

In this section we describe a decomposition of $(X, 0)$ as a union of semi-algebraic subgerms. In Section 6, we will describe it through resolution and we will relate it with the thick-thin decomposition of $(X, 0)$ constructed in [1]. In Part 3, we will show that this decomposition can be recovered using the bilipschitz geometry of $(X, 0)$ and that it determines the geometry of the polar and discriminant of the general plane projection ℓ.

This decomposition will be constructed by starting with a carrousel decomposition of $(\mathbb{C}^2, 0)$ with respect to the discriminant curve of a general plane projection, lifting it to $(X, 0)$, and then performing an amalgamation of some pieces. It is a modification of the decomposition discussed in [1] so we recall the essential details.

Setup. From now on we assume $(X, 0) \subset (\mathbb{C}^n, 0)$ and our coordinates $(z_1 \dots, z_n)$ in $\mathbb{C}^n$ are chosen so that z_1 and z_2 are general linear forms and $\ell := (z_1, z_2) \colon X \to \mathbb{C}^2$ is a general linear projection for X. We denote by Π the polar curve of ℓ and by $\Delta = \ell(\Pi)$ its discriminant curve.

Instead of considering a standard ϵ-ball $\mathbb{B}_\epsilon$ as Milnor ball for $(X, 0)$, we will use, as in [1], a standard "Milnor tube" associated with the Milnor-Lê fibration for the map $h := z_1|_X \colon X \to \mathbb{C}$. Namely, for some sufficiently small ϵ_0 and some $R > 0$ we define for $\epsilon \le \epsilon_0$:

$$B_\epsilon := \{(z_1, \dots, z_n) : |z_1| \le \epsilon, |(z_1, \dots, z_n)| \le R\epsilon\} \quad \text{and} \quad S_\epsilon = \partial B_\epsilon\,,$$

where ϵ_0 and R are chosen so that for $\epsilon \le \epsilon_0$:

(1) $h^{-1}(t)$ intersects the standard sphere $\mathbb{S}_{R\epsilon}$ transversely for $|t| \le \epsilon$;
(2) the polar Π and its tangent cone meet S_ϵ only in the part $|z_1| = \epsilon$.

The existence of such ϵ_0 and R is proved in [1, Section 4].

For any subgerm $(A, 0)$ of $(X, 0)$, we write

$$A^{(\epsilon)} := A \cap S_\epsilon \, \subset \, S_\epsilon\,.$$

In particular, when A is semi-algebraic and ϵ is sufficiently small, $A^{(\epsilon)}$ is the ϵ–link of $(A, 0)$.

Refined carrousel. We now consider a carrousel decomposition of $(\mathbb{C}^2, 0)$ with respect to the germ $(\Delta, 0)$ (see Section 2) and we refine it taking into account very thin zones around branches of Δ (Definition 4.6). Each branch Δ' of Δ lies in some $D(q)$-piece. If a very thin zone about this Δ' is a $D(q')$ with $q' > q$ then we add this very thin zone as a piece. We call this new piece a VT-piece, and the original $D(q)$ piece becomes an $A(q, q')$-piece.

We then lift this refined carrousel decomposition to $(X, 0)$ by ℓ. The inverse image of a piece of any one of the types $B(q)$, $A(q, q')$ or $D(q)$ is a disjoint union of pieces of the same type (from the point of view of its inner geometry). By a "piece" of the decomposition of $(X, 0)$ we will always mean a connected piece, *i.e.*, a connected component of the inverse image of a piece of $(\mathbb{C}^2, 0)$. Notice that the inverse image of a very thin zone about the discriminant may have pieces containing no part of of the polar.

Amalgamation. We now simplify this decomposition of $(X,0)$ by amalgamating certain pieces (this is a similar to [1, Section 13]). We do this in three steps.

(1) *Amalgamating non-empty D-pieces.* Any D-piece which contains part of the polar and is not a VT-piece we amalgamate with the piece which has a common boundary with it.

(2) *Amalgamating empty D-pieces.* Whenever a piece of X is a $D(q)$-piece containing no part of the polar we amalgamate it with the piece which has a common boundary with it. Notice that the amalgamation may form new D-pieces. We continue this amalgamation iteratively until the only D-pieces containing part of the polar are VT pieces; some B-pieces can be amalgamated in this iterative process. Also, $D(1)$ pieces may be created during this process, and are then amalgamated with $B(1)$-pieces. The resulting pieces, which are metrically conical, will still be called $B(1)$-pieces, even though they could have A- or D-topology.

(3) *Amalgamation of A-pieces.* Finally we simplify the decomposition of X further by amalgamating any $A(q,q')$-piece with the piece "outside" it, *i.e.*, the adjacent piece along the boundary with rate q. Note that we do not amalgamate pieces of type $A(q,q)$ ("special annuli" in the terminology of [1], which can arise as lifts of $D(q)$ pieces of the discriminant carrousel).

We can do the same amalgamation first on the carrousel decomposition for the discriminant $\Delta \subset \mathbb{C}^2$ and then lift to X and amalgamate further there as needed.

Example 5.1. Figure 5 shows the carrousel sections for the discriminant of the general plane projection of the singularity D_5 of Example 4.5 followed by the section of the refined carrousel, and then the carrousel after amalgamation.

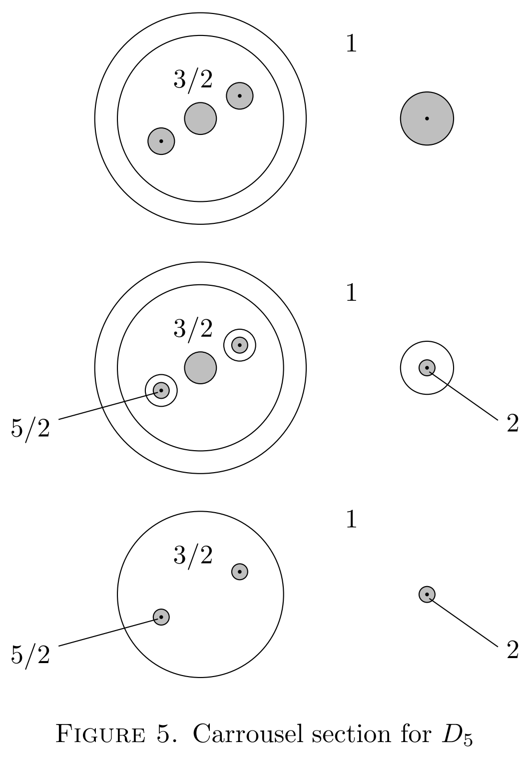


FIGURE 5. Carrousel section for D_5

Note that, except for the $B(1)$-piece, each piece of the decomposition of $(\mathbb{C}^2, 0)$ has one "outer boundary" and some number (possibly zero) of "inner boundaries." When we lift pieces to $(X, 0)$ we use the same terminology *outer boundary* or *inner boundary* for the boundary components of the component of the lifted pieces. Such a piece may have several components to its outer boundary, but they all have the same rate q say, while the inner boundary components then have rates larger than q. The property of having one rate q on all outer boundary components and larger rates on inner boundary components is preserved by the above amalgamation process, so it holds for the pieces after amalgamation (again, excluding the $B(1)$-pieces).

Definition 5.2. The *rate* of a piece of the decomposition of $(X, 0)$ (after the above amalgamation process) is the rate of the outer boundaries of the piece, or is 1 if the piece is an amalgamated $B(1)$.

If $q > 1$ is the rate of some piece of $(X, 0)$ we denote by X_q the union of all pieces of X with this rate q. There is a finite collection of such rates $q_1 > q_2 > \cdots > q_{\nu-1} > 1$. We set $q_\nu = 1$ and denote by X_1 the closure of the complement of the union of the X_{q_i} with $i \le \nu - 1$ (this is also the union of the amalgamated $B(1)$-pieces). Then we can write $(X, 0)$ as the union

$$(X, 0) = \bigcup_{i=1}^{\nu} (X_{q_i}, 0)\,,$$

where the semi-algebraic sets $(X_{q_i}, 0)$ are pasted along their boundary components.

6. Geometric decomposition and resolution

In this section, we describe the geometric decomposition through a suitable resolution of $(X, 0)$. We now consider the minimal good resolution $\pi\colon (\widetilde{X}, E) \to (X, 0)$ with the following properties:

(1) it resolves the basepoints of a general linear system of hyperplane sections of $(X, 0)$;
(2) it resolves the basepoints of the family of polar curves of generic plane projections.

This resolution is obtained from the minimal good resolution of $(X, 0)$ by blowing up further until the basepoints of the two kinds are resolved.

Let Γ be the dual graph of this resolution. As in Section 4 we define a $\mathcal{P}$-node of Γ as a vertex which represents a $\mathcal{P}$-curve. Similarly, an *$\mathcal{L}$-node* in Γ is a vertex which represents an *$\mathcal{L}$-curve*, *i.e.*, an exceptional curve which intersects the strict transform of a general hyperplane section.

A *node* in our resolution graph Γ is one which is either an $\mathcal{L}$-node or a $\mathcal{P}$-node or a vertex having valence ≥ 3 or genus weight > 0. A *chain* is a connected subgraph whose vertices have valence ≤ 2 in Γ and genus weight 0 (we sometimes also include edges incident to the ends of the chain). It is a *string* if none of its vertices are $\mathcal{L}$- or $\mathcal{P}$-nodes, and a *bamboo* if it is a string ending in a vertex of valence 1 (a *leaf*).

Proposition 6.1. *For each $i = 1, \dots, \nu$, let G_i be the subgraph of Γ which consists of the nodes with rate $\ge q_i$ plus maximal strings between them and attached bamboos (so $G_1 = \Gamma$). For each $i = 1 \dots, \nu$, we then have $X_{q_i} = \pi(\overline{\mathcal{N}(G_i) \smallsetminus \mathcal{N}(\bigcup_{j<i} G_j)})$ up to homeomorphism.* $\square$

In particular the graph Γ with nodes weighted by the rates q of the corresponding X_q determines the geometric decomposition of $(X,0)$ into subgerms $(X_{q_i},0)$ up to homeomorphism.

Proof. Let $\rho\colon Y \to \mathbb{C}^2$ be the minimal resolution of Δ which resolves the basepoints of the family of images $\ell(\Pi_{\mathcal{D}})$ by ℓ of the polar curves of generic plane projections. We set $\rho^{-1}(0) = \bigcup_{k=1}^{m} C_k$. Denote by R the resolution graph of ρ. Let V_Δ be the set of vertices of R which represent exceptional curves of $\rho^{-1}(0)$ intersecting the strict transform of a general member of the family. Let V_N be the set of nodes of R, *i.e.*, vertices with valency ≥ 3 taking into account the arrows as edges. We denote by (k_0) the root vertex of R.

Consider the carrousel decomposition of $\mathbb{C}^2$ adapted to Δ, after refinement by very thin zones and amalgamation of the non-empty D-pieces which are not VT-pieces (step (1) in the amalgamation process of Section 5). Then, by construction, the pieces of this carrousel are obtained as follows:

- The $B(1)$-piece is $\rho(\mathcal{N}(C_{k_0}))$.
- The other B-pieces are the $\rho(\mathcal{N}(C_k))$ where $(k) \in V_N$.
- The pieces containing part of Δ are the $\rho(\mathcal{N}(C_k))$ where $(k) \in V_\Delta$.
- The empty D-pieces are the $\rho(\mathcal{N}(C_k))$ where (k) is a leaf which is not a node.
- Each A-piece N is a union $N = \bigcup_{(k)\in\sigma} \rho(N(C_k))$, where σ is a maximal string between two nodes (excluding these nodes). If two nodes are adjacent we blow up first between them to create a string.

We call any vertex of $V_N \cup V_\Delta \cup \{(k_0)\}$ a node of R. In particular each node of R has a rate which is the rate of the corresponding piece in the carrousel.

Let $\pi'\colon X' \to X$ be the Hirzebruch-Jung resolution of $(X,0)$ obtained by pulling back the resolution ρ by the cover ℓ and then normalizing and resolving the remaining cyclic quotient singularities. We denote its dual graph by Γ'. Denote by $\ell'\colon X' \to Y$ the morphism defined by $\ell \circ \pi' = \rho \circ \ell'$. For each node (k) of R denote S_k the set of vertices (i) of Γ' such that ℓ' maps E_i birationally on C_k. Then π' factors through π, each node of Γ' belongs to an S_k and in particular the $\mathcal{L}$-nodes belong to S_{k_0}, while each $\mathcal{P}$-node belongs to an S_k where $k \in N_\Delta$. Moreover each maximal string σ of R lifts to a maximal string in Γ'.

After adjusting the plumbing neighbourhoods $N(E_i)$ if necessary, we deduce from this how each piece of the carrousel is lifted by ℓ: for a node (k) of R and $N = \rho(\mathcal{N}(C_k))$, we have:

$$\ell^{-1}(N) = \bigcup_{(i)\in S_k} \pi'(\mathcal{N}(E_i)),$$

and for a maximal string σ in R and $N = \bigcup_{(k)\in\sigma} \rho(N(C_k))$, we have

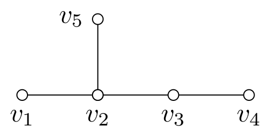
$$\ell^{-1}(N) = \bigcup_{(i)\in\sigma'} \pi'(N(E_i)),$$

where σ' is the lifting of σ in Γ'.

We can blow down Γ' to obtain Γ. In the process some bamboos may be blown down completely, and there can be blowing down on chains joining nodes. When

bamboos disappear it corresponds to amalgamation of empty D-pieces. The interpretation of the amalgamation of the remaining A-pieces determined by the chains in the resolution graph Γ leads to the desired result. $\square$

Example 6.2. Let $(X,0)$ be the D_5 singularity with equation $x^2y+y^4+z^2=0$ as in Example 4.5. The multiplicities of a general linear form h are given by the minimum of the compact part of the three divisors (x), (y) and (z): $(h\circ\pi) = E_1+2E_2+2E_3+E_4+E_5+h^*$. The minimal resolution



resolves a general linear system of hyperplane sections and the strict transform of h is one curve intersecting E_3. So v_3 is the single $\mathcal{L}$-node and the geometric decomposition is:

$$X = X_{5/2}\cup X_2\cup X_{3/2}\cup X_1,$$

where $X_{5/2}$ and X_2 are the two polar zones. Figure 6 shows Γ with vertices v_i weighted with the corresponding rates q_i. We have $X_{5/2} = \pi(\mathcal{N}(E_6))$, $X_2 =$

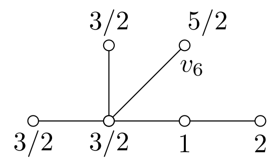


Figure 6. Decomposition of D_5

$\pi(\mathcal{N}(E_4))$, $X_1 = \pi(N(E_3))$ and $X_{3/2} = \pi(\overline{N(E_1\cup E_2\cup E_5)\smallsetminus N(E_3)})$.

Remark 6.3. In [1] we proved the existence and unicity of a decomposition $(X,0) = (Y,0)\cup(Z,0)$ of a normal complex singularity $(X,0)$ into a thick part $(Y,0)$ and a thin part $(Z,0)$. The thick part is essentially the metrically conical part of $(X,0)$ with respect to the inner metric while the thin part shrinks faster than linearly in size as it approaches the origin. It follows immediately from [1] that the geometric decomposition $(X,0) = \bigcup(X_{q_i},0)$ is in fact a refinement of this thick-thin decomposition. In particular, the thick part contains $(X_1,0)$.

Part 3: Analytic invariants from Lipschitz geometry

7. Resolving a general linear system of hyperplane sections

In this section we prove:

Theorem 7.1 (Theorem 1.2 in the Introduction)**.** *If $(X,0)$ is a normal complex surface singularity then the outer Lipschitz geometry on X determines:*

1. *the decorated resolution graph of the minimal good resolution of $(X,0)$ which resolves the basepoints of a general linear system of hyperplane sections;*
2. *the multiplicity of $(X,0)$ and the maximal ideal cycle in its resolution;*
3. *for a general hyperplane H, the outer geometry of the curve $(X\cap H,0)$.*

We need some preliminary results about the relationship between thick-thin decomposition and resolution graph and about the usual and metric tangent cones.

We will use the Milnor balls B_ϵ for $(X,0) \subset (\mathbb{C}^n,0)$ defined at the beginning of Section 5, which are associated with the choice of a general linear plane projection $\ell = (z_1,z_2)\colon \mathbb{C}^n \to \mathbb{C}^2$, and we consider again the generic linear form $h := z_1|_X\colon X \to \mathbb{C}$. We denote by $A^{(\epsilon)} = A \cap B_\epsilon$ for ϵ sufficiently small the link of any semi-algebraic subgerm $(A,0) \subset (X,0)$.

Thick-thin decomposition and resolution graph. According to [1], the thick-thin decomposition of (X,0) is determined up to homeomorphism close to the identity by the inner geometry, and hence by the outer geometry (specifically, the homeomorphism $\phi\colon (X,0) \to (X,0)$ satisfies $d(p,\phi(p)) \le ||p||^q$ for some $q > 1$). It is described through resolution as follows. The thick part $(Y,0)$ is the union of semi-algebraic sets which are in bijection with the $\mathcal{L}$-nodes of any resolution which resolves the basepoints of a general linear system of hyperplane sections. More precisely, let us consider the minimal good resolution $\pi\colon (\widetilde{X},E) \to (X,0)$ of this type and let Γ be its resolution graph. Let $(1),\dots,(r)$ be the $\mathcal{L}$-nodes of Γ. For each $i = 1,\dots,r$ consider the subgraph Γ_i of Γ consisting of the $\mathcal{L}$-node (i) plus any attached bamboos (ignoring $\mathcal{P}$-nodes). Then the thick part is given by:

$$(Y,0) = \bigcup_{i=1}^{r} (Y_i,0) \quad \text{where} \quad Y_i = \pi(N(\Gamma_i)).$$

Let $\Gamma'_1,\dots,\Gamma'_s$ be the connected components of $\Gamma \setminus \bigcup_{i=1}^r \Gamma_i$. Then the thin part is:

$$(Z,0) = \bigcup_{j=1}^{s} (Z_j,0) \quad \text{where} \quad Z_j = \pi(\mathcal{N}(\Gamma'_j))\,.$$

We call the links $Y_i^{(\epsilon)}$ and $Z_j^{(\epsilon)}$ of the thick and thin pieces *thick and thin zones* respectively.

Inner geometry and $\mathcal{L}$-nodes. Since the graph Γ_i is star-shaped, the thick zone $Y_i^{(\epsilon)}$ is a Seifert piece in a graph decomposition of the link $X^{(\epsilon)}$. Therefore the inner metric already tells us a lot about the location of $\mathcal{L}$-nodes: for a thick zone $Y_i^{(\epsilon)}$ with unique Seifert fibration (*i.e.*, not an $\mathbb{S}^1$-bundle over a disk or annulus) the corresponding $\mathcal{L}$-node is determined in any negative definite plumbing graph for the pair $(X^{(\epsilon)}, Y_i^{(\epsilon)})$. However, a thick zone may be a solid torus $\mathbb{D}^2 \times \mathbb{S}^1$ or toral annulus $I \times \mathbb{S}^1 \times \mathbb{S}^1$. Such a zone corresponds to a vertex on a chain in the resolution graph and different vertices along the chain correspond to topologically equivalent solid tori or toral annuli in the link $X^{(\epsilon)}$. Thus, in general inner metric is insufficient to determine the $\mathcal{L}$-nodes and we need to appeal to the outer metric.

Tangent cones. We will use the Bernig-Lytchak map $\phi\colon \mathcal{T}_0(X) \to T_0(X)$ between the metric tangent cone $\mathcal{T}_0(X)$ and the usual tangent cone $T_0(X)$ ([3]). We will need its description as given in [1, Section 9].

The linear form $h\colon X \to \mathbb{C}$ restricts to a fibration $\phi_j\colon Z_j^{(\epsilon)} \to \mathbb{S}^1_\epsilon$, and, as described in [1, Theorem 1.7], the components of the fibers have inner diameter $o(\epsilon^q)$ for some $q > 1$. If one scales the inner metric on $X^{(\epsilon)}$ by $\frac{1}{\epsilon}$ then in the Gromov-Hausdorff limit as $\epsilon \to 0$ the components of the fibers of each thin zone collapse to points, so each thin zone collapses to a circle. On the other hand, the rescaled

thick zones are basically stable, except that their boundaries collapse to the circle limits of the rescaled thin zones. The result is the link $\mathcal{T}^{(1)}X$ of the so-called metric tangent cone $\mathcal{T}_0X$ (see [1, Section 9]), decomposed as

$$\mathcal{T}^{(1)}X = \lim_{\epsilon\to 0}{}^{GH}\frac{1}{\epsilon}X^{(\epsilon)} = \bigcup \mathcal{T}^{(1)}Y_i\,,$$

where $\mathcal{T}^{(1)}Y_i = \lim^{GH}_{\epsilon\to 0}\frac{1}{\epsilon}Y_i^{(\epsilon)}$, and these are glued along circles.

One can also consider $\frac{1}{\epsilon}X^{(\epsilon)}$ as a subset of $S_1 = \partial B_1 \subset \mathbb{C}^n$ and form the Hausdorff limit in S_1 to get the link $T^{(1)}X$ of the usual tangent cone T_0X (this is the same as taking the Gromov-Hausdorff limit for the outer metric). One thus sees a natural branched covering map $\mathcal{T}^{(1)}X \to T^{(1)}X$ which extends to a map of cones $\phi\colon \mathcal{T}_0(X) \to T_0(X)$ (first described in [3]).

We denote by $T^{(1)}Y_i$ the piece of $T^{(1)}X$ corresponding to Y_i (but note that two different Y_i's can have the same $T^{(1)}Y_i$).

Proof of (1) *of Theorem 1.2.* Let L_j be the tangent line to Z_j at 0 and $h_j = h|_{\partial L_j\cap B_\epsilon}\colon \partial(L_j\cap B_\epsilon) \overset{\cong}{\to} \mathbb{S}^1_\epsilon$. We can rescale the fibration $h_j^{-1}\circ\phi_j$ to a fibration $\phi'_j\colon \frac{1}{\epsilon}Z_j^{(\epsilon)} \to \partial(L_j\cap B_1)$, and written in this form ϕ'_j moves points distance $O(\epsilon^{q-1})$, so the fibers of ϕ'_j are shrinking at this rate. In particular, once ϵ is sufficiently small the outer Lipschitz geometry of $\frac{1}{\epsilon}Z_j^{(\epsilon)}$ determines this fibration up to homotopy, and hence also up to isotopy, since homotopic fibrations of a 3-manifold to S^1 are isotopic (see e.g., [7, p. 34]).

Consider now a rescaled thick piece $M_i = \frac{1}{\epsilon}Y_i^{(\epsilon)}$. The intersection $L_i \subset M_i$ of M_i with the rescaled link of the curve $\{h=0\} \subset (X,0)$ is a union of fibers of the Seifert fibration of M_i. The intersection of a Milnor fiber of h with M_i gives a homology between L_i and the union of the curves along which a Milnor fiber meets ∂M_i, and by the previous paragraph these curves are discernible from the outer geometry, so the homology class of L_i in M_i is known. It follows that the number of components of L_i is known and L_i is therefore known up to isotopy, at least in the case that M_i has unique Seifert fibration. If M_i is a toral annulus the argument still works, but if M_i is a solid torus we need a little more care.

If M_i is a solid torus it corresponds to an $\mathcal{L}$-node which is a vertex of a bamboo. If it is the leaf of this bamboo then the map $\mathcal{T}^{(1)}Y_i \to T^{(1)}Y_i$ is a covering. Otherwise it is a branched covering branched along its central circle. Both the branching degree p_i and the degree d_i of the map $\mathcal{T}^{(1)}Y_i \to T^{(1)}Y_i$ are determined by the Lipschitz geometry, so we can compute d_i/p_i, which is the number of times the Milnor fiber meets the central curve of the solid torus M_i. A tubular neighbourhood of this curve meets the Milnor fiber in d_i/p_i disks, and removing it gives us a toral annulus for which we know the intersection of the Milnor fibers with its boundary, so we find the topology of $L_i \subset M_i$ as before.

We have thus shown that the Lipschitz geometry determines the topology of the link $\bigcup L_i$ of the strict transform of h in the link $X^{(\epsilon)}$. Denote $L'_i = L_i$ unless (M_i, L_i) is a knot in a solid torus, *i.e.*, L_i is connected and M_i a solid torus, in which case put $L'_i = 2L_i$ (two parallel copies of L_i). The resolution graph we are seeking represents a minimal negative definite plumbing graph for the pair $(X^{(\epsilon)}, \bigcup L'_i)$, for $(X,0)$. By [15] such a plumbing graph is uniquely determined by the topology. When decorated with arrows for the L_i only, rather than the L'_i, it

gives the desired decorated resolution graph Γ. So Γ is determined by $(X,0)$ and its Lipschitz geometry. □

Proof of (2) *of Theorem 1.2.* Recall that $\pi\colon \widetilde{X} \to X$ denotes the minimal good resolution of $(X,0)$ which resolves a general linear system of hyperplane sections. Denote by h^* the strict transform by π of the general linear form h and let $\bigcup_{k=1}^d E_k$ the decomposition of the exceptional divisor $\pi^{-1}(0)$ into its irreducible components. By point (1) of the theorem, the Lipschitz geometry of $(X,0)$ determines the resolution graph of this resolution and also determines $h^* \cdot E_k$ for each k. We therefore recover the total transform $(h) := \sum_{k=1}^d m_k(h)E_k + h^*$ of h (since $E_l \cdot (h) = 0$ for all $l = 1,\dots,d$ and the intersection matrix $(E_k \cdot E_l)$ is negative definite).

In particular, the maximal ideal cycle $\sum_{i=1}^d m_i kh)E_k$ is determined by the geometry, and the multiplicity also, since it is given by $\sum_{k=1}^d m_k(h)E_k \cdot h^*$. □

Proof of (3) *of Theorem 1.2.* The argument is similar to that of the proof of Proposition 3.1, but considering a continuous arc γ inside the conical part of $(X,0)$ (for the inner metric) with the property that $d(0,\gamma(t)) = O(t)$. Then for k sufficiently small, the intersection $X \cap B(\gamma(t), kt)$ consists of $\mu = \mathrm{mult}(X,0)$ 4-balls $D_1^4(t),\dots,D_\mu^4(t)$ with $d(D_j^4(t), D_k^4(t)) = O(t^{q_{jk}})$ before we change the metric by a bilipschitz homeomorphism. The q_{jk} are determined by the outer geometry of $(X,0)$ and are still determined after a bilipschitz change of the metric by the same argument as the last part of the proof of 3.1. The q_{jk} do not depend on the choice of the arc γ and if one chooses γ inside a general hyperplane section $H \cap X$, Proposition 3.1 asserts that they are the Puiseux exponents of the branches and coincidence exponents between branches of $\ell'(H \cap X)$. By Lemma 3.3 they determine the embedded topological type of this plane curve, and then also its Lipschitz geometry by Remark 3.2. Pham and Teissier's result in [16] mentioned earlier, that the restriction $\ell'\colon H \cap X \to \ell'(H \cap X)$ is bilipschitz for the outer geometry, finishes the proof. □

8. Detecting the decomposition

In this section we prove that the decomposition $(X,0) = \bigcup_{i=1}^\nu (X_{q_i},0)$ of Sections 5 and 6 can be recovered (up to small deformation) using the semi-algebraic Lipschitz outer geometry of X. Specifically, we will prove:

Theorem 8.1. *The outer semi-algebraic Lipschitz geometry of* $(X,0)$ *determines a decomposition of* $(X,0)$ *into semi-algebraic subgerms* $(X,0) = \bigcup_{i=1}^\nu (X'_{q_i},0)$ *glued along their boundaries, where each* $\overline{X'_{q_i} \smallsetminus X_{q_i}}$ *is a union of collars of type* $A(q_i,q_i)$. *So* X'_{q_i} *is obtained from* X_{q_i} *by adding an* $A(q_i,q_i)$ *collar on each outer boundary component of* X_{q_i} *and removing an* $A(q_j,q_j)$ *collar at each inner boundary component* (q_j *being the rate for the boundary component).*

We start with a rough outline of the argument. First a definition:

Definition 8.2 (Normal embedding). A semi-algebraic germ $(Z,0) \subset (\mathbb{C}^n,0)$ is *normally embedded* if its inner and outer metrics agree up to bilipschitz equivalence.

Our proof of Theorem 8.1 consists in discovering the polar zones in the germ $(X,0)$ by exploring them with horn neighbourhoods of real arcs. The key point is that for such a horn neighbourhood with rate q' intersecting a polar zone with rate q, the intersection with X either has non-trivial topology or is not normally embedded if $q' < q$ (Lemma 8.11) and is normally embedded with trivial topology

if $q' > q$. We use this to construct the pieces $(X'_{q_i}, 0)$ inductively for decreasing q_i. The details need a little care. We start with some definitions and constructions.

Definition 8.3 (***q*-neighbourhood).** Let $(U,0) \subset (\widehat{X},0) \subset (X,0)$ be semi-algebraic sub-germs. A *q-neighbourhood of $(U,0)$ in $(\widehat{X},0)$* is a germ $(N,0) \supseteq (U,0)$ with $N \subseteq \widehat{X} \cap \{x \in X \mid d(x,U) \le K d(x,0)^q\}$ for some K, using inner metric in $(\widehat{X},0)$. An *outer q-neighbourhood* is defined similarly, using outer metric.

Definition 8.4 (**Smoothing).** The outer boundary of X_{q_i} attaches to $A(q',q_i)$ pieces of the (non-amalgamated) carrousel decomposition, so we can add $A(q_i,q_i)$ collars to the outer boundary of X_{q_i} to obtain a q_i-neighbourhood of X_{q_i} in $X \setminus \bigcup_{j<i} X_{q_j}$. We use $X^+_{q_i}$ to denote such an enlarged version of X_{q_i}. An arbitrary q_i-neighbourhood of X_{q_i} in $X \setminus \bigcup_{j<i} X_{q_j}$ can be embedded in one of the form $X^+_{q_i}$, and we call this process *smoothing*.

This smoothing process is topologically determined up to homeomorphism since a q_i-neighbourhood of the form $X^+_{q_i}$ of X_{q_i} in $X \setminus \bigcup_{j<i} X_{q_j}$ is characterized up to homeomorphism among all q_i-neighbourhoods of X_{q_i} in $X \setminus \bigcup_{j<i} X_{q_j}$ by the fact that all boundary components of its link are tori and the number of boundary components is minimal among q_i-neighbourhoods of X_{q_i} in $X \setminus \bigcup_{j<i} X_{q_j}$. Smoothing can be applied more generally to a q-neighbourhood of any subgerm $(Y,0)$ of $(X,0)$ whose outer boundary components are inner boundary components of $A(q',q)$ pieces with $q' < q$.

Definition 8.5 (**(Q,C)-admissible arc).** Let Q be a finite set of positive rationals and $C > 0$. An arc $\gamma\colon [0,1) \to \mathbb{C}^n$ with $\gamma(0) = 0$ is called a *(Q,C)-admissible arc* if it can be expressed in the form $\gamma(t) = (z_1(t), \dots, z_n(t))$ where each $z_j(t)$ is a Puiseux polynomial whose exponents are in Q with all coefficients bounded by C and if $||\gamma(t)|| = O(t)$ (in particular, $1 \in Q$).

Remark 8.6. Notice that a real analytic arc on $(X,0)$ can be expressed by Puiseux series expansions, but in general these are not Puiseux polynomials. This is why a (Q,C)-admissible arc is defined as an arc in $(\mathbb{C}^n,0)$ rather than in $(X,0)$. But given a real analytic arc on $(X,0)$, one can approximate it by a (Q,C)-admissible arc in $(\mathbb{C}^n,0)$ as close as we need by truncating its Puiseux series expansions.

Definition 8.7. We say two semi-algebraic germs $(A,0)$ and $(B,0)$ *intersect* if 0 is an isolated point of $A \cap B$.

A *component* of a semi-algebraic germ $(A,0)$ means the germ of the closure of a connected component of $(A \cap B_{\epsilon_0}) \setminus \{0\}$ for sufficiently small ϵ_0 (*i.e.*, the family of B_ϵ with $\epsilon \le \epsilon_0$ should be a family of Milnor balls for A).

Definition 8.8 (**Connected (a,q)-horn about an arc).** Let $(\widehat{X},0) \subset (X,0)$ be a pure 4-dimensional real semi-algebraic subgerm. Let $\gamma\colon [0,1) \to (\mathbb{C}^n,0)$ be a (Q,C)-admissible arc. For $a \in \mathbb{R}_+$ and $q \in \mathbb{Q}$ with $q \ge 1$ we define the *(a,q)-horn about γ* as the germ of the set

$$\mathcal{H}(\gamma;a,q) := \bigcup_{t\in[0,1)} B(\gamma(t), a|\gamma(t)|^q)\,,$$

where $B(x,r)$ denotes the ball of radius r around $x \in \mathbb{C}^n$ and we require $a < 1$ if $q = 1$.

$\widehat{X} \cap \mathcal{H}(\gamma; a, q)$ may just be $\{0\}$ or may have one or more components. In the latter case we define a *connected* (a, q)*-horn in* $(\widehat{X}, 0)$ as the germ of a component of the germ $\widehat{X} \cap \mathcal{H}(\gamma; a, q)$. We simply write $(\mathcal{H}_c(\gamma; a, q, \widehat{X}), 0)$ for a connected (a, q)-horn, even though there may be several of them for the given data. Finally, we say that this connected horn is a *disk horn* if it has a q-neighbourhood N in $\widehat{X}$ such that the intersection $N \cap S_\epsilon$ is a disk (then necessarily of size $O(\epsilon^q)$) for all sufficiently small ϵ.

Sketch. We call *real slice* of a real algebraic germ $(Z, 0) \subset (\mathbb{C}^n, 0)$ the intersection of Z with with $\{h = t\} \cap P$ where h is a general linear form and P a general $(2n-1)$-plane in $\mathbb{C}^n \cong \mathbb{R}^{2n}$. In order to get a first flavour of what will happen in the sequel of the section, we will now vizualize some connected horns by drawing the real slices of the involved germs $(\widehat{X}, 0)$, $(\mathcal{H}(\gamma; a, q), 0)$ and then of $(\mathcal{H}_c(\gamma; a, q, \widehat{X}), 0)$. Let us use again the notations of the previous section: $\ell = (z_1, z_2)$ is a general plane projection for $(X, 0)$ and $h = z_1|_X$. We consider a $B(q_1)$-piece B of $(X, 0)$ which is not a polar zone. We assume that $q_1 > 1$ and that the sheets of the cover $\ell|_X \to C^2$ inside this zone have pairwise contact $> q_1$, *i.e.*, there exists $q_2 > q_1$ such that for an arc γ_0 in B, all the arcs $\gamma_0' \neq \gamma_0$ in $\ell^{-1}(\ell(\gamma_0))$ have contact at most q_2 with γ and the contact q_2 is reached for at least one of these arcs.

Figure 7 represents some real slices of connected horns $\mathcal{H}_c(\gamma; a, q, X)$ for some arcs γ in $(\mathbb{C}^n, 0)$ which are close to B *i.e.*, such that there is an $\mathcal{H}_c(\gamma; a, q)$ inside B with $q > q_1$.

The dotted circle represents the boundary of the real slice of the horn $\mathcal{H}(\gamma; a, q)$. The real slices of the connected horns are the thickened arcs.

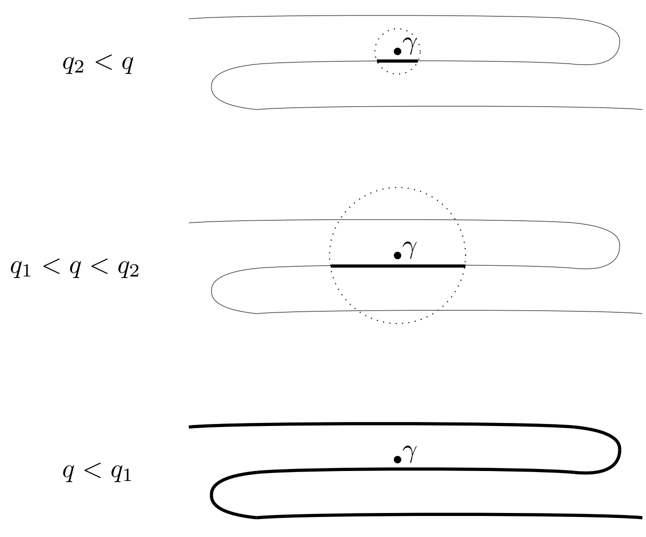


FIGURE 7. real slices of horns

Let us now take γ close to a polar zone N, $a > 0$ and $q > 0$ such that $q_1 < q < q_2$. Figure 8 represents the corresponding connected horn for $\widehat{X} = X$ and for $\widehat{X} = \overline{X \setminus N}$.

From now on we often suppress in our notation the dependence of objects on (Q, C) and the fact that we are always dealing with germs at 0.

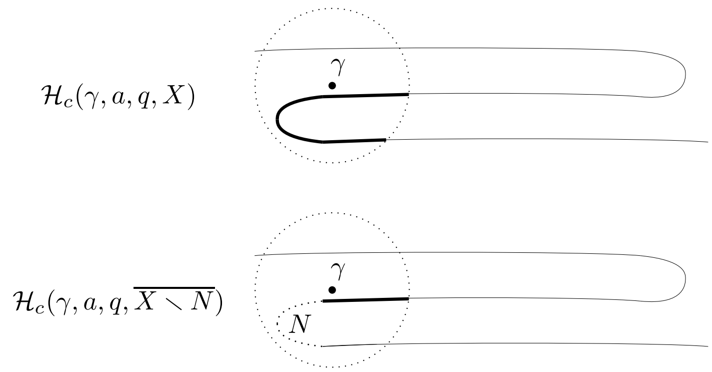


FIGURE 8.

Definition 8.9. A connected (a, q)-horn $\mathcal{H}_c(\gamma; a, q, \widehat{X})$ is *extremal* if

(1) each $\mathcal{H}_c(\gamma'; a', q', \widehat{X})$ with $q' > q$ contained in $\mathcal{H}_c(\gamma; a, q, \widehat{X})$ is a normally embedded disk horn;
(2) no $\mathcal{H}_c(\gamma; a', q', \widehat{X})$ with $q' < q$ containing $\mathcal{H}_c(\gamma; a, q, \widehat{X})$ is a normally embedded disk horn.

We say that an extremal $\mathcal{H}_c(\gamma; a, q, \widehat{X})$ is *minimal* if it also satisfies:

(3) There is a q-neighbourhood of $\mathcal{H}_c(\gamma; a, q, \widehat{X})$ in $\widehat{X}$ such that no larger q-neighbourhood in $\widehat{X}$ deformation retracts onto a q'-neighbourhood in $\widehat{X}$ of an extremal $\mathcal{H}_c(\gamma'; a', q', \widehat{X})$ contained in it with $q' > q$. Here we allow γ' to be (Q_1, C)-admissible for some Q_1 with $Q_1 \supseteq Q$.

For $\widehat{X} \subset X$ as above and any $q \geq 1$ and $a > 0$ define

$$Z_{q,a}(\widehat{X}) := \bigcup \{\mathcal{H}_c(\gamma; a, q, \widehat{X}) \mid \gamma \text{ is } (Q, C)\text{-admissible and } \mathcal{H}_c(\gamma; a, q, \widehat{X}) \text{ is minimal extremal}\}\,.$$

Proposition 8.10. *There exist (Q', C') and $a > 0$ such that for any (Q, C) with $Q \supseteq Q'$ and $C \geq C'$ the following holds using (Q, C)-admissible horns:*

Set $X^{(1)} = X$. The largest q such that $Z_{q,a}(X^{(1)})$ is nonempty is q_1. Define

$$Z_{q_1,a} := Z_{q_1,a}(X^{(1)})\,.$$

Then $Z_{q_1,a}$ is a q_1-neighbourhood of X_{q_1}, so let $Z^+_{q_1,a}$ be a smoothing of it and define

$$X^{(2)} := \overline{X^{(1)} \setminus Z^+_{q_1,a}}\,.$$

Inductively, if $Z^+_{q_i,a}$ and $X^{(i+1)} := \overline{X^{(i)} \setminus Z^+_{q_i,a}}$ have been constructed and $q_i > 1$ then the largest q with $q < q_i$ such that $Z_{q,a}(X^{(i+1)}) \neq \emptyset$ is q_{i+1}. Moreover

$$Z_{q_{i+1},a} := Z_{q_{i+1},a}(X^{(i+1)})$$

is a q_{i+1}-neighbourhood of $X_{q_{i+1}} \cap X^{(i+1}$, so it has a smoothing $Z^+_{q_{i+1},a}$ and we define

$$X^{(i+2)} := \overline{X^{(i+1)} \setminus Z^+_{q_{i+1},a}}\,.$$

This process ends in ν steps with $q_\nu = 1$.

To prove Proposition 8.10 we first need some notation and a couple of Lemmas. Denote by $\widehat{X}_{q_i}$ any set of the form

$$\widehat{X}_{q_i} := X \smallsetminus \bigcup_{j<i} X^+_{q_j}$$

(see Definition 8.4). In particular, assuming the correctness of Proposition 8.10, the set $X^{(i)}$ is of the form $\widehat{X}_{q_i}$.

Let $\pi\colon (\widetilde{X}, E) \to (X, 0)$ be the minimal good resolution which resolves the base-points of the family of polars of general linear projections (as in Section 4).

Lemma 8.11. *Let N^+ be a q_i-neighbourhood of a polar zone $N = \pi(\mathcal{N}(E_i))$ of rate q_i, where $E_i \subset E$ is a $\mathcal{P}$-curve in E. Assume that the outer boundary of N is connected. Then for every $\mathcal{H}_c(\gamma; a, q', \widehat{X}_{q_i})$ with $q' < q_i$ which intersects N^+, no $H_c(\gamma; a', q', \widehat{X}_{q_i})$ with $a' > a$ is normally embedded.*

Proof. For simplicity of notation we take $N^+ = N$ (the proof is the same for N^+).

We can adjust N slightly so that it is a branched cover of its image $\ell(N)$ and $\ell(N)$ is a piece of the carrousel decomposition with rate q_i (before doing amalgamations). Then $\ell(N)$ has an $A(q'', q_i)$ annulus outside it for some q'', which we will simply call $A(q'', q_i)$. The lift of $A(q'', q_i)$ by ℓ is a covering space. Denote by $\tilde{A}(q'', q_i)$ the component of this lift that intersects N; it is connected since the outer boundary of N is connected. Therefore $\tilde{A}(q'', q_i)$ is contained in a $N(\Gamma')$ where Γ' is a $\mathcal{P}$-Tjurina component of the resolution graph Γ so we can apply part (1) of Lemma 4.8 to any suitable arc inside it. This will be the key argument later in the proof.

The restriction of ℓ to a neighbourhood of a branch of the polar curve is a double cover, so the covering $N \to \ell(N)$ has degree > 1. Hence the restriction to its outer boundary has degree > 1, so the covering $\tilde{A}(q'', q_i) \to A(q'', q_i)$ has degree > 1.

We will prove the lemma for q' with $q'' < q' < q_i$ since it is then certainly true for smaller q'. Choose p' with $q' < p' < q_i$

Let γ_0 be a smooth arc $\gamma_0\colon [0,1] \to \ell(N \cap \mathcal{H}_c(\gamma; a, q', \widehat{X}_{q_i})) \subset \mathbb{C}^2$ with $||\gamma_0(t)|| = O(t)$ as $t \to 0$. Consider the function

$$\gamma_s(t) := \gamma_0(t) + (0, st^{p'}) \quad \text{for } (s,t) \in [0,1]^2 .$$

We can think of this as a family, parametrized by s, of arcs $t \mapsto \gamma_s(t)$ or as a family, parametrized by t, of real curves $s \mapsto \gamma_s(t)$. For t sufficiently small $\gamma_1(t)$ lies in $\ell(\mathcal{H}_c(\gamma; a', q', \widehat{X}_{q_i}))$ and also lies in the $A(q'', q_i)$ annulus mentioned above. Note that for any s the point $\gamma_s(t)$ is distance $O(t)$ from the origin.

We now take two different continuous lifts $\gamma_s^{(1)}(t)$ and $\gamma_s^{(2)}(t)$ by ℓ of the family of arcs $\gamma_s(t)$, for $0 \le s \le 1$, with $\gamma_0^{(1)} = \gamma$ and $\gamma_0^{(2)}$ also in N (possible by the previous comment on covering degree $\tilde{A}(q'', q_i) \to A(q'', q_i)$). To make notation simpler we set $P_1 = \gamma_1^{(1)}$ and $P_2 = \gamma_1^{(2)}$. Since the points $P_1(t)$ and $P_2(t)$ are on different sheets of the covering of $A(q'', q_i)$, a shortest path between them will have to travel through N, so its length $l_{inn}(t)$ satisfies $l_{inn}(t) = O(t^{p'})$.

We now give a rough estimate of the outer distance $l_{out}(t) = ||P_1(t) - P_2(t)||$ which will be sufficient to show $\lim_{t\to 0}(l_{out}(t)/l_{inn}(t)) = 0$, proving the lemma. For this, we choose p'' with $p' < p'' < q_i$ and consider the arc $p\colon [0,1] \to \mathbb{C}^2$ defined by:

$$p(t) := \gamma_{s_t}(t) = \gamma_0(t) + (0, t^{p''}) \quad \text{with } s_t := t^{p''-p'} ,$$

and its two liftings $p_1(t) := \gamma^{(1)}_{s_t}(t)$ and $p_2(t) := \gamma^{(2)}_{s_t}(t)$, belonging to the same sheets of the cover ℓ as the arcs P_1 and P_2. A real slice of the situation is represented in Figure 9.

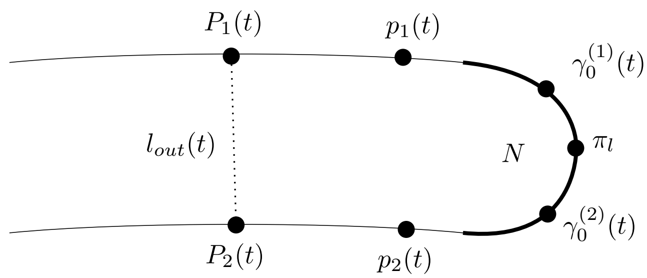


Figure 9.

The points $p_1(t)$ and $p_2(t)$ are inner distance $O(t^{p''})$ apart by the same argument as before, so their outer distance is at most $O(t^{p''})$. By Lemma 4.8 the line from $p(t)$ to $\gamma_1(t)$ lifts to almost straight lines which are almost parallel, from $p_1(t)$ to $P_1(t)$ and from $p_2(t)$ to $P_2(t)$ respectively, with degree of parallelism increasing as $t \to 0$. Thus as we move from the pair $p_i(t)$, $i = 1, 2$ to the pair $P_i(t)$ the distance changes by $f(t)(t^{p'} - t^{p''})$ where $f(t) \to 0$ as $t \to 0$. Thus the outer distance $l_{out}(t)$ between the pair is at most $O(t^{p''}) + f(t)(t^{p'} - t^{p''})$. Dividing by $l_{inn}(t) = O(t^{p'})$ gives $l_{out}(t)/l_{inn}(t) = O(f(t))$, so $\lim_{t\to 0}(l_{out}(t)/l_{inn}(t)) = 0$, as desired. $\square$

Notice that Lemma 8.11 can give the polar rate q_i in simple examples such as the following.

Example 8.12. Assume $(X, 0)$ is a hypersurface with equation $z^2 = f(x, y)$ where f is reduced. The projection $\ell = (x, y)$ is general and its discriminant curve Δ has equation $f(x, y) = 0$. Consider a branch δ of Δ which lifts to a polar zone N in X. We consider the Puiseux expansion of δ truncated just after its last characteristic exponent:

$$y = \sum_{i=1}^{n} a_i x^{q_i}.$$

The rate of N is the minimal $s \geq q_n$ such that for any small λ, the curve δ': $y = \sum_{i=1}^{n} a_i x^{q_i} + \lambda x^s$ is in $\ell(N)$. In order to compute s, we set $s' = st_n$ where t_n is the denominator of q_n and we parametrize δ' as:

$$x = w^{t_n}, \quad y = \sum_{i=1}^{n} a_i w^{t_n q_i} + w^{s'}$$

Replacing in the equation, and approximating by elimination of the monomials with higher order in s', we obtain $z^2 \sim a w^{s'+b}$ for some $a \neq 0$ and some positive integer b, giving an outer distance of $O(w^{\frac{s'+b}{2}})$ between the two sheets of the projection ℓ. So the optimal s' such that the curve δ' is in $\ell(N)$ is given by $\frac{s'+b}{2} = s'$, *i.e.*, $s' = b$, so $s = b/t_n$.

We apply this to the singularity D_5 with equation $z^2 = -(x^2 y + y^4)$ (Example 4.5). The discriminant curve of $\ell = (x, y)$ has equation $y(x^2 + y^3) = 0$. For the polar

zone $\pi(\mathcal{N}(E_6))$, which projects on a neighbourhood of the cusp $\delta_2 = \{x^2+y^3=0\}$, we use $y = w^2$, $x = iw^3 + w^{s'}$, so $z^2 \sim -2iw^{5+s'}$, so this polar zone has rate $5/2$. Similarly one computes that the polar zone $\pi(\mathcal{N}(E_4))$, which projects on a neighbourhood of $\delta_1 = \{y = 0\}$, has rate 2.

Definition 8.13. If $(A, 0)$ is a semi-algebraic germ and $\gamma\colon [0,1] \to A$ a real analytic arc with $\gamma(0) = 0$, we say γ has *degree of contact q with A* if $\mathcal{H}(\gamma; q, a) \cap A = \{0\}$ as a germ for all $q' > q$ and all $a > 0$ and $\mathcal{H}(\gamma; q, a) \cap A \neq \{0\}$ for all $q' < q$ and all a. We say that an arc γ is *close to the inner boundary of $\widehat{X}_{q_i}$* if it has degree of contact q with some inner boundary component of $\widehat{X}_{q_i}$ with q greater than or equal to the rate of that boundary component.

Lemma 8.14. *If γ is an arc with degree of contact $> q_i$ with some inner boundary component of $\widehat{X}_{q_i}$, denote by $\tilde{q}$ this contact degree; otherwise set $\tilde{q} = q_i$. So $\tilde{q} \geq q_i$.*

(1) *If γ is not close to the inner boundary of $\widehat{X}_{q_i}$ and $q' > \tilde{q}$ then any $\mathcal{H}_c(\gamma; a, q', \widehat{X}_{q_i})$ is a normally embedded disk horn.*
(2) *If γ has degree of contact $q < q_i$ with X_{q_i} and $q < q'$ then any $\mathcal{H}_c(\gamma; a, q', \widehat{X}_{q_i})$ is a normally embedded disk horn outside X_{q_i}.*
(3) *If $q' < \tilde{q}$ and $\tilde{q} > q_i$ then no $\mathcal{H}_c(\gamma; a, q', \widehat{X}_{q_i})$ is a normally embedded disk horn.*

Proof. (1). We assume γ is not close to the inner boundary and $q' > \tilde{q}$. Then $\mathcal{H}_c(\gamma; a, q', \widehat{X}_{q_i})$ is strictly inside $\widehat{X}_{q_i}$, in the sense that $\mathcal{H}_c(\gamma; a, q', \widehat{X}_{q_i})$ equals $\mathcal{H}_c(\gamma; a, q', X)$ and no arc in $\mathcal{H}_c(\gamma; a, q', \widehat{X}_{q_i})$ is close to the inner boundary.

Assume first that the strict transform of some arc γ' in $\mathcal{H}_c(\gamma; a, q', \widehat{X}_{q_i})$ meets a $\mathcal{P}$-curve E_ν of $E = \pi^{-1}(0)$ in a point $x \in E_\nu$. Since γ' is not close to the inner boundary of $\widehat{X}_{q_i}$, the corresponding polar zone must be in $\widehat{X}_{q_i}$, and hence has rate $\leq q_i$. Thus q' is larger than the rate of the polar zone, so the strict transform of any arc in $\mathcal{H}_c(\gamma; a, q', \widehat{X}_{q_i})$ meets E_ν at x. Then by part (2) of Lemma 4.8, the tangent spaces to X along the arcs have the same limit P_x in the Grassmannian $G(n, 2)$. Therefore $H_c(\gamma, a, q', \widehat{X}_{q_i})$ is asymptotically flat as t tends to 0, so it is a normally embedded disk horn.

If the strict transforms of arcs in $\mathcal{H}_c(\gamma; a, q', \widehat{X}_{q_i})$ do not meet a $\mathcal{P}$-curve then they are inside a $\pi(N(\Gamma'))$ where Γ' is a subgraph of the resolution graph of π which does not contain any $\mathcal{P}$-node. Then, by part (1) of Lemma 4.8, $H_c(\gamma, a, q', \widehat{X}_{q_i})$ is a normally embedded disk horn.

(2). As $q < q'$ and $q < q_i$ then any $\mathcal{H}_c(\gamma; a, q', \widehat{X}_{q_i})$ is automatically outside X_{q_i}. The proof is now as before: $\mathcal{H}_c(\gamma; a, q', \widehat{X}_{q_i})$ is strictly inside $\widehat{X}_{q_i}$ and if an arc in $\mathcal{H}_c(\gamma; a, q', \widehat{X}_{q_i})$ is in a polar zone, the polar zone has rate $\leq q$, so the strict transforms of all arcs of $\mathcal{H}_c(\gamma; a, q', \widehat{X}_{q_i})$ meet the $\mathcal{P}$-curve at the same point. Otherwise all arcs of $\mathcal{H}_c(\gamma; a, q', \widehat{X}_{q_i})$ are inside a $\pi(N(\Gamma'))$ as above.

(3). If $\tilde{q} > q_i$ then $\widehat{X}_{q_i}$ has non-empty inner boundary and γ has degree of contact $\tilde{q}$ with this boundary. Since $q' < \tilde{q}$, $\mathcal{H}_c(\gamma; a, q', \widehat{X}_{q_i})$ includes meridian circles in this boundary, so $\mathcal{H}_c(\gamma; a, q', \widehat{X}_{q_i})$ has non-trivial topology and cannot be a disk horn. $\square$

Proof of Proposition 8.10. We can expand all branches of the discriminant Δ of the general plane projection $\ell|_X \colon X \to \mathbb{C}^2$ (where $\ell = (z_1, z_2)$ as before) as Puiseux

series of the form $(u, \sum_i a_i u^{p_i})$ and then we take Q' to contain all the Puiseux exponents in these expansions up to the largest which are essential to the geometry (so including characteristic Puiseux exponents of the branches, polar zone rates and coincidence exponents between branches). Consider a carrousel decomposition of $\mathbb{C}^2$ with respect to Δ. Each B_κ of the carrousel with $\kappa = (f, p_k)$ is saturated by complex curves having Puiseux polynomial expansions obtained by varying the coefficient of the last term x^{p_k} in these Puiseux expansions. The same is true for D-pieces. We consider the lifts to X of all these curves (over all B_κ and all D-pieces) and take the constant C' to be the largest absolute value of a coefficient of a term x^q with $q \in Q'$ occurring in the Puiseux expansions of one of these curves. Let $\gamma(u) = (z_1(u), z_2(u), z_3(u), \dots, z_3(u))$ be the Puiseux expansion of one of these liftings γ. Then the complex curve γ is saturated by real analytic arcs γ_θ with Puiseux expansions $t \in \mathbb{R} \mapsto \gamma(te^{i\theta})$. Let $\tilde{q}$ be the largest exponent in Q'. For each θ, consider the arc $\widehat{\gamma}_\theta$ in $\mathbb{C}^n$ whose Puiseux expansion is obtained by truncating all the terms with exponents $\geq \tilde{q}$. Then γ_θ is inside a connected horn $\mathcal{H}_c(\widehat{\gamma}_\theta; a, \tilde{q})$. Therefore, we have shown that for any $Q \supseteq Q'$ and $C \geq C'$ any piece (before the amalgamation process) of the carrousel, except maybe A-pieces, is contained in a union of $(a, \tilde{q})$-connected horns centered at (Q, C)-admissible arcs.

With the above choices of Q and C we have:

Lemma 8.15. *Let $a > 0$. Each X_{q_i} has a q_i-neighbourhood $\widetilde{X}_{q_i}(a)$ containing every $\mathcal{H}_c(\gamma; a, q_i, \widehat{X}_{q_i})$ which is centered at a (Q, C)-admissible arc γ and which intersects some q_i-neighbourhood of X_{q_i}.*

Proof. We will deal with X_{q_i} component by component, so to simplify notation we assume $X_{q_i} \setminus \{0\}$ is connected. Then the image $\ell(X_{q_i})$ is a subset of $(\mathbb{C}^2, 0)$ consisting of a carrousel piece B_κ, possibly with smaller pieces attached on its inside boundary; in particular, it lies within the outer boundary of B_κ. Here $\kappa = (f, q_i)$, where $f = \sum_{j=1}^{k-1} a_j x^{p_j}$ is a Puiseux polynomial for which there is a Puiseux series

$$y = \sum_{j=1}^{k-1} a_j x^{p_j} + a_k x^{q_i} + \dots$$

describing some component of the discriminant Δ. A complex curve in a q_i-neighbourhood of B_κ must have the form $(x, \sum_{i=1}^{k} a_i x^{p_i} + b x^{q_i} + o(x^{q_i}))$ for some b.

Let γ be a (Q, C)-admissible arc such that some $\mathcal{H}_c(\gamma; a, q_i, \widehat{X}_{q_i})$ intersects a q_i-neighbourhood X'_{q_i} of X_{q_i}. Then the arc $\ell \circ \gamma$ lies in a q_i-neighbourhood N of B_κ. By enlarging N is necessary, one can assume that N is saturated by complex curves. Let $(\ell \circ \gamma)(t) = e^{i\theta}(t, z_2(t))$ be a Puiseux expansion of $\ell \circ \gamma$. Then the complex curve with Puiseux expansion $(x, z_2(x))$ must lie in N so $(\ell \circ \gamma)(t)$ has the form $e^{i\theta}(t, \sum_{i=1}^{k} a_i t^{p_i} + b t^{q_i} + g(t))$ with $g(t)$ a Puiseux polynomial of the form $g(t) = \sum_{p > q_i} b_p t^p$. Since b and the coefficients b_p are bounded by C, we can then choose $N = N(C)$ where $N(C)$ is the q_i-neighbourhood of B_κ saturated by complex curves $(x, \sum_{i=1}^{k} a_i x^{p_i} + b x^{q_i} + \sum_{p > q_i} b_p x^p)$ with coefficients b and b_p bounded by C. It is independent of γ. Now we consider the enlarged neighbourhood $N(C + a)$, *i.e.* taking the coefficients b and b_p bounded by $C + a$. Then $N(C + a)$ contains $\ell(\mathcal{H}_c(\gamma; a, q_i, \widehat{X}_{q_i}))$. We can then take for $\widetilde{X}_{q_i}$ the component of the lift of $N(C+a)$ by $\ell|_X$ which contains X_{q_i}. $\square$

We now continue with the proof of Proposition 8.10. We fix $a > 0$ and $Q \supseteq Q'$ and $C \geq C'$. Let $X^{(1)} = X$.

By part (1) of Lemma 8.14 there is no extremal $\mathcal{H}_c(\gamma; a, q, X^{(1)})$ with $q > q_1$ and an $\mathcal{H}_c(\gamma; a, q_1, X^{(1)})$ which is extremal must have γ in a q_1-neighbourhood of X_{q_1}. By Lemma 8.15, this q_1-neighbourhood $\widetilde{X}_{q_1}$ can be chosen independent of γ.

We now claim that any $\mathcal{H}_c(\gamma; a, q_1, X^{(1)})$ contained in $\widetilde{X}_{q_1}$ is minimal extremal. Indeed, it is extremal since if $\mathcal{H}_c(\gamma; a, q, X^{(1)})$ has $q > q_1$ then it is a normally embedded disk horn by part (1) of Lemma 8.14, while if $q < q_1$ then it is not normally embedded if Lemma 8.11 applies, and not a disk horn otherwise. Finally, it is minimal since no $\mathcal{H}_c(\gamma'; a, q, X^{(1)})$ with $q > q_1$ is extremal.

Since X_{q_1} is contained in a union of connected (a, q_1)-horns centered at (C, Q)-admissible arcs, the set

$$Z_{q_1,a} = \bigcup \{\mathcal{H}_c(\gamma; a, q_1, X^{(1)}) \mid \mathcal{H}_c(\gamma; a, q_1, X^{(1)}) \text{ is minimal extremal}\}$$

is a q_1-neighbourhood of X_{q_1}.

This completes the start of the induction in Proposition 8.10.

For the induction step we assume $Z_{q_j,a}$ is constructed with the right properties for $j < i$ (in particular, that the q_j's are the numbers occurring in the carrousel decomposition) and we consider $X^{(i)} = \overline{X^{(i-1)} \smallsetminus Z^{+}_{q_{i-1},a}}$. The inductive step now only considers horns with rate $q' < q_{i-1}$.

Let γ be a (Q, C)-admissible arc. We will show:

(1) There is no minimal extremal horn $\mathcal{H}_c(\gamma; a, q', X^{(i)})$ with $q_i < q' < q_{i-1}$;
(2) if $q' = q_i$ and $\mathcal{H}(\gamma; a, q_i)$ does not intersect any q_i-neighbourhood of X_{q_i} then no $\mathcal{H}_c(\gamma; a, q', X^{(i)})$ is extremal;
(3) if $q' = q_i$ and γ has contact q_i with any inner boundary of X_{q_i} and there is a $\mathcal{H}_c(\gamma; a, q')$ intersecting X_{q_i}, then $\mathcal{H}_c(\gamma; a, q', X^{(i)})$ is minimal extremal.

We prove these three statements below. Assuming them, we again see that

$$Z_{q_i,a} := \bigcup \{\mathcal{H}_c(\gamma; a, q_i, X^{(i)}) \mid \mathcal{H}_c(\gamma; a, q_i, X^{(i)}) \text{ is minimal extremal}\}$$

is a q_i-neighbourhood of $X_{q_i} \cap X^{(i)}$, so Proposition 8.10 is then proven.

Proof of item (1). Assume $H = H_c(\gamma; a, q', X^{(i)})$ with $q_i < q' < q_{i-1}$. If γ has degree of contact $> q_i$ with some inner boundary component of X_{q_i}, denote by $\tilde{q}$ this contact degree; otherwise set $\tilde{q} = q_i$. If $q_i < q' < \tilde{q}$ then H is not a normally embedded disk horn by part (3) of Lemma 8.14 and it remains so if q' is increased slightly, so H is not extremal.

If γ is close to the inner boundary then $\tilde{q} \geq q_{i-1}$, so $q' < \tilde{q}$ and we are in the above situation. So if $q' > \tilde{q}$ then γ is not close to the inner boundary, and H is a normally embedded disk horn by part (1) of Lemma 8.14. It remains so if q' is decreased slightly, so it is not extremal.

Finally, suppose $q' = \tilde{q}$. Then since $q' > q_i$, q' is exactly the degree of contact of γ with the inner boundary, so we can find a q'-neighbourhood of H such that each cross section is a disk with a hole coming from the inner boundary of $X^{(i)}$. Choosing a γ' closer to the inner boundary, say with contact q'' with $q' > q'' > q_i$ we can find a q''-neighbourhood of an $\mathcal{H}_c(\gamma'; a, q'', X^{(i)})$ which is a deformation retract of H. Thus H is not minimal.

Proof of item (2). We have $q' = q_i$ and γ is outside every q_i-neighbourhood of X_{q_i}. In this case $\mathcal{H}(\gamma; a, q', X^{(i)})$ is a normally embedded disk horn by part (2) of

Lemma 8.14 and remains so when q' is decreased slightly, so $\mathcal{H}(\gamma; a, q', X^{(i)})$ is not extremal.

Proof of item (3). Assume $q' = q_i$ and $\mathcal{H}_c(\gamma; a, q')$ intersects X_{q_i}. Then $\mathcal{H}_c(\gamma; a, q', X^{(i)})$ is extremal since increasing q' gives a normally embedded disk horn by part (1) of Lemma 8.14 while decreasing q' gives a horn with non-trivial topology if the inner boundary of the relevant component of X_{q_i} has non-empty inner boundary, or a non-normally embedded horn if the inner boundary is empty (Lemma 8.11).

Some q' neighbourhood H of $\mathcal{H}(\gamma; a, q', X^{(i)})$ in $X^{(i)})$ spans the narrow direction of the component of $X_{q_i} \cap X^{(i)}$ intersecting it. If that component has no inner boundaries then no horn inside H with larger q' is extremal, while if there are inner boundaries then a horn inside H with larger q' cannot include all the topology of the component.

This completes the proof of items (1)–(3) and hence of Proposition 8.10. □

Proof of Theorem 8.1. Proposition 8.10 implies Theorem 8.1 for the outer metric on X, using using $X'_{q_i} := Z^+_{q_i,a}$. The semi-algebraic bilipschitz invariance of the process described in Proposition 8.10 is given by the following proposition. □

Proposition 8.16. *Let $(X, 0) \subset (\mathbb{C}^n, 0)$ and $(X', 0) \subset (\mathbb{C}^{n'}, 0)$ be two germs of normal complex surfaces endowed with the outer metrics (X, d) and (X', d'). Assume that there is a semi-algebraic bilipschitz map $\Phi\colon (X, 0) \to (X', 0)$. Then the inductive process described in Proposition 8.10 leads to the same sequence of rates q_i for both $(X, 0)$ and $(X', 0)$ and for $a > 0$ sufficiently large, the corresponding sequences of subgerms $Z_{q_i,a}$ in X and $Z'_{q_i,a}$ in X' have the property that $\Phi(Z_{q_i,a})$ and $Z'_{q_i,a}$ coincide after moving their boundaries by addition or removal of collars of type $A(q, q)$.*

Proof. The proof follows from the following four observations.

(1). Given (Q, C) and (arbitrarily large) q there exists (Q', C') such that if H is a connected q-horn in X centered at a (Q, C)-admissible arc in $(\mathbb{C}^n, 0)$, then $\Phi(H)$ is in a q-horn centered at a (Q', C')-admissible arc in $(\mathbb{C}^{n'}, 0)$.

(2). Let K be the bilipschitz constant of Φ in a fixed neighbourhood of the origin. Let γ be a (Q, C)-admissible arc. The image $\Phi(\mathcal{H}(\gamma; a, q)) \cap X'$ is in general not the intersection of a q-horn with X', but there exists Q', C' and a constant $K' > K^{q+1}$ so that for any such γ there exists a (Q', C')-horn γ' in $\mathbb{C}^{n'}$ such that

$$\Phi(\mathcal{H}(\gamma; a, q)) \cap X \subset \mathcal{H}(\gamma'; aK', q) \cap X' \,,$$

The same inclusions hold for connected horns.

(3). The property of being normally embedded for a subgerm is invariant under bilipschitz maps. Moreover, the property of being extremal for a connected horn $\mathcal{H}_c(\gamma; a, q)$ is independent of sufficiently large a. Therefore, $\mathcal{H}_c(\gamma; a, q, X)$ is extremal if and only if the corresponding $\mathcal{H}_c(\gamma'; aK', q, X')$ of point (2) is extremal and the same holds for "minimal extremal".

(4). Let $(U, 0) \subset (\widehat{X}, 0) \subset (X, 0)$ be semi-algebraic sub-germs. If $(N, 0)$ is a q-neighbourhood of $(U, 0)$ in $(\widehat{X}, 0)$ then $(\Phi(N), 0)$ is a q-neighbourhood of $(\Phi(U), 0)$ in $(\Phi(\widehat{X}), 0)$. □

9. Carrousel and polar from Lipschitz geometry

The aim of this section is to complete the proof of Theorem 1.3.

Let $\ell = (z_1, z_2)\colon X \to \mathbb{C}^2$ be the general plane projection and $h = z_1|_X$ the general linear form chosen in Section 7. We denote by $F = h^{-1}(\epsilon)$ the Milnor fibre of h.

Proposition 9.1. *The outer Lipschitz geometry of $(X, 0)$ determines:*

1. *the carrousel sections, including the images of the polar zones (see Section 5), inside $\ell(F)$ about each tangent line to the discriminant curve Δ of ℓ,*
2. *the number of components of Δ in each B- or D-piece of the carrousel decomposition,*
3. *the number of components of the polar curve Π in each B- or D-piece of X.*

Proof. By Theorem 8.1 we may assume that we have recovered the pieces X_{q_i} of X for $i = 1, \ldots, \nu$ (up to addition/deletion of $A(q, q)$-type annular collars at the boundaries).

We will recover the carrousel sections by an inductive procedure on the rates q_i starting with q_1: we first recover the innermost pieces, *i.e.*, the pieces of $\ell(X_{q_1} \cap F)$. Then we glue to their outer boundaries the pieces corresponding to $\ell((X_{q_1} \cup X_{q_2}) \cap F) \smallsetminus \ell(X_{q_1} \cap F)$, and so on. The outer geometry will determine at each step the shape of the new pieces we have to glue and how they are glued. The procedure is illustrated in the Examples 9.2, 9.3 and 9.4 after this proof.

For any $i < \nu$ we will denote by X''_{q_i} the result of adding $A(q', q_i)$ collars to all components of the outer boundary $\partial_0 X_{q_i}$ of X_{q_i} for some q' with $q_{i+1} < q' < q_i$. We can assume that the added annular pieces are chosen so that the map ℓ maps each by a covering map to a $A(q', q_i)$ piece in $\mathbb{C}^2$. We will denote by $\partial_0 X''_{q_i}$ the outer boundary of X''_{q_i}, so it is a horn-shaped cone with rate q' on a family of circles. We denote

$$F_{q_i} := F \cap X_{q_i}, \quad F''_{q_i} := F \cap X''_{q_i}\,,$$

and their outer boundaries by

$$\partial_0 F_{q_i} = F \cap \partial_0 X_{q_i}, \quad \partial_0 F''_{q_i} = F \cap \partial_0 X''_{q_i}\,.$$

A straightforward consequence of Theorem 1.2 is that the outer geometry determines the isotopy class of the Milnor fibre F and then also that of the intersections $F_{q_i} = F \cap X_{q_i}$ for $i < \nu$.

Set $B_{q_1} := \ell(F_{q_1})$, so B_{q_1} consists of the innermost disks of the carrousel section we wish to recover. Consider the restriction $\ell\colon F_{q_1} \to B_{q_1}$. We want to show that this branched cover can be seen in terms of the outer geometry of a neighbourhood of X_{q_1}. Since B_{q_1} is a union of disks, the same is true for $B''_{q_1} := \ell(F''_{q_1})$. The argument of Lemma 8.11 implies that the outer geometry of $\partial_0 F''_{q_1}$ determines up to isotopy the restriction $\partial_0 F''_1 \to \partial_0 B''_{q_1}$, and hence the degree of the branched covering map from each component of F''_{q_1} to its image disk, and also determines which components of F''_{q_1} lie over each component of B''_{q_1}. Since F''_{q_1} is just F_{q_1} with collars on its boundary, the same is true for F_{q_1} and B_{q_1}.

We restrict now to a connected component $F^0_{q_1}$ of F_{q_1} and consider the branched cover $\ell|_{F^0_{q_1}} : F^0_{q_1} \to \ell(F^0_{q_1})$. As described above, its degree n is determined by the outer geometry. We wish to determine the number k of branch points of this map. Since the cover is general, the inverse image by $\ell_{|F^0_{q_1}}$ of a branch point consists of $n-1$ points in $F^0_{q_1}$ and Hurwitz formula applied to $\ell_{|F^0_{q_1}}$ gives:

$$k = n - \chi(F^0_{q_1})\,.$$

This k is the number of branches of the polar curve Π meeting $F^0_{q_1}$. So if $X^0_{q_1}$ denotes the component of X_{q_1} containing $F^0_{q_1}$, then k is also the number of branches of Π inside $X^0_{q_1}$ (note that the number of components of $X^0_{q_1} \cap F$ is the denominator of q_1). Since the outer geometry tells which components of F_{q_1} lie over each component of B_{q_1}, we see how many components of the discriminant Δ meet each component of B_{q_1}. This completes the initial step of our induction.

Overlapping and nesting. In the next steps $i \geq 2$, we give special attention to the following two phenomena. It can happen that $\ell(X_{q_i})$ contains connected components of some $\ell(X_{q_j})$ for $j < i$. We say that X_{q_i} *overlaps* X_{q_j}. This phenomenon is illustrated further in Example 9.3 where the thick part X_1 overlaps the components of a thin zone. When $i > 2$ a new phenomenon that we call *nesting* may occur (which is why we treat first the step $i = 2$ and then the following steps $i > 2$): it could happen that X_{q_i} has common boundary components with both X_{q_k} and $X_{q_{k'}}$ with $k' < k < i$ and that the corresponding components of X_{q_k} overlap those of $X_{q_{k'}}$. The image by ℓ of some inner boundary components of X_{q_i} with rate $q_{k'}$ will then be nested in some with rate q_k. An example of such a nesting is given in 9.4.

We now consider F_{q_2}. As before, we know from the outer metric for F''_{q_2} how the outer boundary $\partial_0 F_{q_2}$ covers its image. We will focus first on a single component $F^0_{q_2}$ of F_{q_2}. The map $\ell|_{F^0_{q_2}} : F^0_{q_2} \to \ell(F^0_{q_2})$ is a covering map in a neighbourhood of its outer boundary, whose degree, m say, is determined from the outer geometry. The image $\ell(\partial_0 F^0_{q_2})$ of the outer boundary is a circle which bounds a disk $B^0_{q_2}$ in the plane $\{z_1 = \epsilon\}$. The image $\ell(F^0_{q_2})$ is this $B^0_{q_2}$, possibly with some smaller disks removed, depending on whether and how X_{q_2} overlaps X_{q_1}. The image of the inner boundaries of $F^0_{q_2}$ will consist of disjoint circles inside $B^0_{q_2}$ of size proportional to ϵ^{q_1} (we say briefly "with rate q_1"). Consider the components of F_{q_1} which fit into the inner boundary components of $F^0_{q_2}$. Their images form a collection of disjoint disks $B_{q_1,1}, \dots, B_{q_1,s}$. For each j denote by $F_{q_1,j}$ the union of the components of F_{q_1} which meet $F^0_{q_2}$ and map to $B_{q_1,j}$ by ℓ. By the first step of the induction we know the degree m_j of the map $\partial_0 F_{q_1,j} \to \partial_0 B_{q_1,j}$. This degree may be less than m, in which case $\ell^{-1}(B_{q_1,j}) \cap F^0_{q_2}$ must consist of $m - m_j$ disks. Thus, after removing $\sum_j (m - m_j)$ disks from $F^0_{q_2}$, we have a subset $\hat{F}^0_{q_2}$ of $F^0_{q_2}$ which maps to $\overline{B^0_{q_2} \setminus \bigcup_{j=1}^s B_{q_1,j}}$ by a branched covering. Moreover, the branch points of the branched cover $\hat{F}^0_{q_2} \to \overline{B^0_{q_2} \setminus \bigcup_{j=1}^s B_{q_1,j}}$ are the intersection points of $F^0_{q_2}$ with the polar Π. We again apply the Hurwitz formula to discover this number: it is $m(1-s) - \chi(\hat{F}^0_{q_2}) = m(1-s) - \chi(F^0_{q_2}) + \sum_{j=1}^s (m - m_j)$.

So as before, we find the number of branches of the polar in each component of X_{q_2}, and since outer geometry tells us which components of F_{q_2} lie over which components of $\ell(F_{q_2})$, we recover how many components of the discriminant meet each component of $\ell(F_{q_2})$.

Different components $F^0_{q_2}$ of F_{q_2} may correspond to the same $B^0_{q_2}$, but as already described, this is detected using outer geometry. We write B_{q_2} for the union of the disks $B^0_{q_2}$ as we run through the components of F_{q_2}. The resulting embedding of B_{q_1} in B_{q_2} is the next approximation to the carrousel.

Iterating this procedure for F_{q_i} with $i > 2$, we now take into account the possible occurrence of nesting. We again focus on one component $F^0_{q_i}$ of F_{q_i}, whose image will be a disk $B^0_{q_i}$ possibly with some subdisks removed. The images of the inner

boundaries of $F^0_{q_i}$ will be circles in $B^0_{q_i}$ of rates q_k with $k < i$. It can now occur that pairs of such circles are nested in each other, with a q_k circle inside the $q_{k'}$ one with $k < k' < i$. Consider an outermost circle S of these nested families of such circles and the disk $\tilde{B}^0$ which it bounds, and a component $\tilde{F}^0$ of $\ell^{-1}(\tilde{B}^0) \cap F^0_{q_i}$ which has an inner boundary component nested inside S. The map $\tilde{F}^0 \to \tilde{B}^0$ given by the restricting ℓ has no branch points, so it is a local homeomorphism away from its boundary, and the argument of Lemma 8.11 implies that its sheets are very close together at its outer boundary. It can thus be picked out using the outer metric. We remove all such pieces from $F^0_{q_i}$ and call the result $F^*_{q_i}$. The situation is now similar to the case $i = 2$ so we re-use some symbols; if S_j, $j = 1, \dots, s$ are the outer nested circles in $B^0_{q_i}$, m_j the degree of ℓ on the part of the inner boundary of $F^*_{q_i}$ which lies over S_j, and m is the degree of ℓ on the outer boundary of $F^*_{q_i}$, then we again get that the number of intersection points of Π with $F^0_{q_i}$ is $m(1-s) - \chi(\hat{F}^*_{q_i}) = m(1-s) - \chi(F^*_{q_i}) + \sum_{j=1}^{s}(m - m_j)$. Since $\chi(F^*_{q_i})$ is computed as the difference of $\chi(F^0_{q_i})$ and the sum of Euler characteristics of the pieces of the form $\tilde{F}^0$ that were removed from $F^0_{q_i}$, the number of intersection points is determined by the geometry.

In this way we iteratively build up the picture of the carrousel section while finding the number of components of the polar in each component of X_{q_i} and therefore the number of components of the discriminant in the pieces of the carrousel. □

Proof of Theorem 1.3. Part (3) of Theorem 1.3 is already proved in the Introduction as a straightforward consequence of (2), which itself is a straightforward consequence of Theorem 9.1. So we must just prove part (1). We have just shown that the outer geometry determines the decomposition of $(X, 0)$ as the union of subgerms $(X_{q_i}, 0)$ as well as the number of components of the polar in each piece. If we consider the decomposition of the link $X^{(\epsilon)}$ as the union of the Seifert fibered manifolds $X^{(\epsilon)}_{q_i}$ then the link L of the polar is a union of Seifert fibers in some of the pieces. The Seifert fibration $X^{(\epsilon)}_{q_i}$ is determined by the rate q_i (in most cases it is, in fact, unique). By taking two parallel copies of L (we will call it $2L$), as in the proof of (1) of Theorem 1.2, we fix the Seifert fibrations and can therefore recover the dual graph Γ we are seeking as the minimal negative plumbing diagram for the pair $(X^{(\epsilon)}, 2L)$. Since we know the number of components of the polar in each piece, the proof is complete. □

Example 9.2. Let us show how the procedures described in Proposition 8.10 and in (1) of Proposition 9.1 work for the singularity D_5, *i.e.*, recover the decomposition $X = X_{5/2} \cup X_2 \cup X_{3/2} \cup X_1$, the carrousel sections and the topology of the discriminant curve Δ. It starts by considering high q for which $Z_{q,a}(X) = \emptyset$. Then, by decreasing q one reconstructs the sequence $X_{q_1}, \dots, X_{q_\nu}$, up to collars as described in Theorem 8.1, using the procedure of Proposition 8.10 (so X_{q_i} is recovered up to collars as $Z^+_{q_i,a}$). At each step we will do the following:

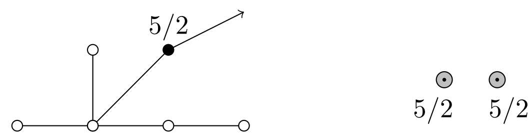
(1) On the left in the figures below we draw the dual resolution graph with black vertices corresponding to the zone X_{q_i} just discovered, while the vertices corresponding to the previous steps are in gray, and the remaining ones in white. We weight each vertex by the corresponding rate q_i.
(2) We describe the cover $\ell \colon F_{q_i} \to B_{q_i}$ and B_{q_i} (see (1) of Proposition 9.1) and on the right we draw the new pieces of the carrousel section. The $\mathcal{D}$-pieces

are in gray. In the carrousel section, we write the rate q_i at each step. Note that only the rates of the B-pieces are relevant for the topological type of the discriminant curve Δ. The rates of $\mathcal{D}$-pieces play no role here.

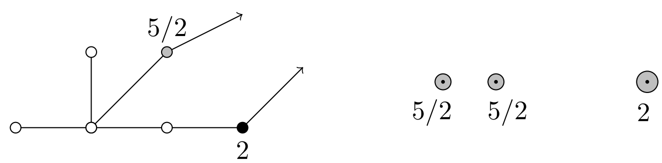
(3) We compute the number of branch points inside the new pieces of the carrousel. We represent them by black points in the carrousel section and we add arrows on the graph corresponding to the strict transform of the corresponding branches of the polar.

We now carry out these steps. We use the indexing $v_1, \dots, v_6$ of the vertices of Γ introduced in Example 4.5.

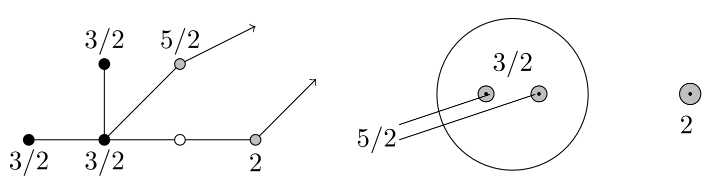
- $q_1 = 5/2$ is the largest q such that $Z_{q_1,a} \neq \emptyset$. We get $X_{5/2} = \pi(\mathcal{N}(E_6))$ up to collars. Using the multiplicities of the general linear form (determined by (2) of Theorem 1.2) we obtain that $F_{5/2} = F \cap \pi(\mathcal{N}(E_6))$ consists of two discs. Then $B_{5/2} = \ell(F_{5/2})$ also consists of two discs. By the Hurwitz formula, each of them contains one branch point of ℓ.



- $q_2 = 2$ is the next q such that $Z_{q,a} \neq \emptyset$ and $X_2 = \pi(\mathcal{N}(E_4))$. F_2 is a disk as well as B_2. Then it creates another carrousel section disk. There is one branching point inside B_2.

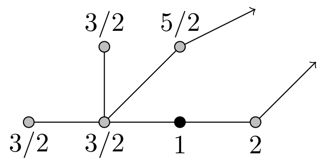


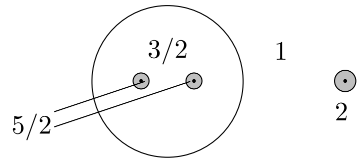
- Next is $q_3 = 3/2$. $F_{3/2}$ is a sphere with 4 holes. Two of the boundary circles are common boundary with F_1, the two others map on the same circle $\partial B_{5/2}$. A simple observation of the multiplicities of the generic linear form shows that there $X_{3/2}$ does not overlap $X_{5/2}$ so $B_{3/2} = \ell(F_{3/2})$ is connected with one outer boundary component and inner boundary glued to $B_{5/2}$.



- Last is $q_4 = 1$, which corresponds to the thick piece.

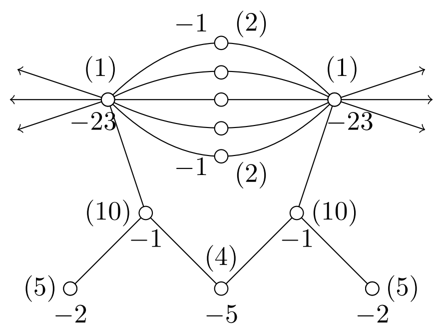
We read in the carrousel section the topology of the discriminant curve Δ from this carrousel sections: one smooth branch and one transversal cusp with Puiseux exponent $3/2$.



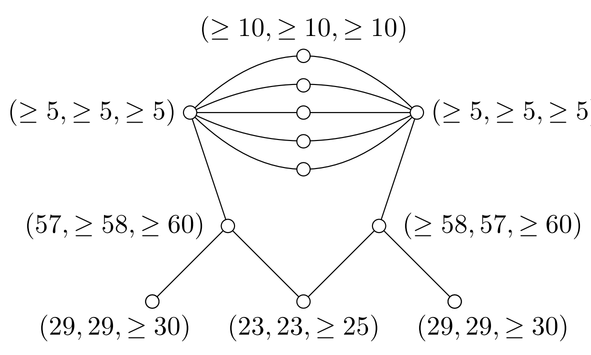


Example 9.3. We now perform the procedure on a more sophisticated example: the surface singularity $(X,0)$ with equation $(zx^2+y^3)(x^3+zy^2)+z^7=0$. This example has been already studied partially in [1] Section 15, and we refer to that paper for details.

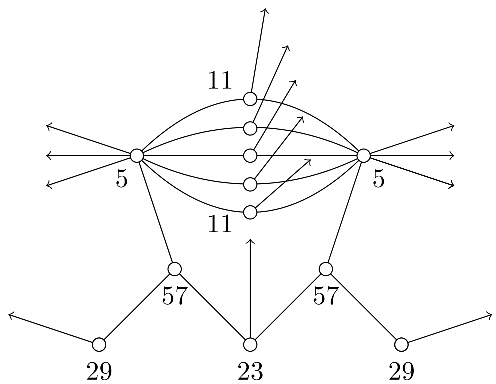
Let us first compute some invariants from the equation: the following picture shows the resolution graph of a general linear system of hyperplane sections. The negative numbers are self-intersection of the exceptional curves while numbers in parentheses are the multiplicities of a generic linear form.



Set $f(x,y,z)=(zx^2+y^3)(x^3+zy^2)+z^7$. The multiplicities of the partial derivatives $(m(f_x),m(f_y),m(f_z))$ are given by:



The basepoints of the family of polars of general plane projections are resolved by this resolution and the strict transform of the general polar Π_ℓ is given by the following graph. The multiplicities are those of a general linear combination $g=af_x+bf_y+cf_z$.



We now compute the rate s of the two polar zones represented by the vertices weighted by 29. Let v_1 be one of these two vertices and v_2 the neighbour node. Let E_1 and E_2 be the respective curves in the exceptional divisor of the resolution $\pi\colon \widetilde{X} \to X$. Consider the very thin zone $N = \pi(\mathcal{N}(E_1))$. Adjusting N if necessary, we can assume that the restriction $\ell|_N$ double covers its image.

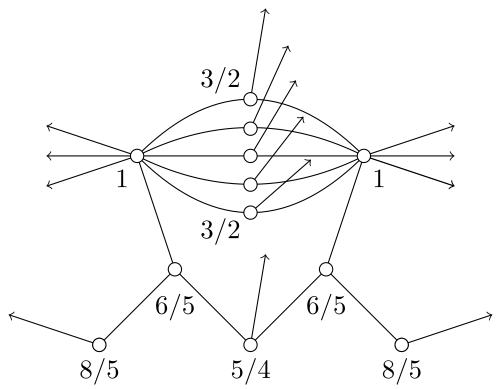
Let γ_1 be a complex curve in N such that its strict transform γ_1^* by π is a curvette of E_1. As the rate of the neighbour zone is $6/5$, the curve $\ell(\gamma_1)$ intersects a carrousel section $z_1 = \epsilon$ in 5 points. As the multiplicity of the generic linear form $z_1|_X$ equals 5 along E_1, it means that γ_1 intersect the Milnor fiber $z_1 = \epsilon$ in 5 points in N and that the restriction $\ell|_{\gamma_1}$ is then 1-to-1. Now let γ_2 be a curve in $\pi(\mathcal{N}(E_2))$ such that γ_2^* is a curvette of E_2. As the multiplicity of $z_1|_X$ along E_2 equals 10 while $\ell(\gamma_2)$ intersects the carrousel section in 5 points, we see that the restriction $\ell|_{\gamma_2}$ double covers its image.

Let us now choose γ_1 and γ_2 so that their links L_1 and L_2 lie in the boundary of the solid torus $N^{(\epsilon)} = N \cap B_\epsilon$. As the self intersection E_1^2 equals -2 in $\tilde{X}$, the intersection $L_1.L_2$ equals 2 on T. Then by the previous arguments the intersection $\ell(L_1) \cdot \ell(L_2)$ equals 2 on the torus $\ell(T)$. In a basis $(M, L) =$ (meridian, longitude) of $H_1(\ell(T); \mathbb{Z})$, the two curves L_1 and L_2 express as : $\ell(L_1) = L + 6M$ and $\ell(L_2) = L + pM$. Therefore $\ell(L_1).\ell(L_2) = p - 6$, which leads to $p = 8$. So we obtain that the contact s between two generic complex curves in $\ell(N)$ equals $8/5$.

We now describe how one would reconstruct the decomposition $X = \bigcup X_{q_i}$ and the carrousel section from the outer geometry using the procedure. The sequence of rates q_i defined in Proposition 8.10 appears in the following order:

$$q_1 = 8/5, q_2 = 3/2, q_3 = 5/4, q_4 = 6/5, q_5 = 1.$$

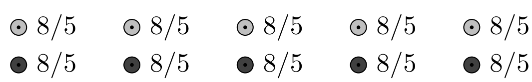
We first indicate on the resolution graph the rate corresponding to each vertex.



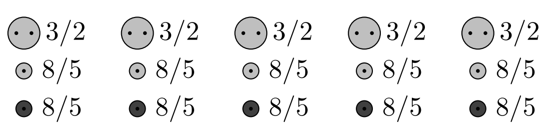
Overlapping. The inverse image by ℓ of the carrousel section of $\ell(X_{5/4})$ consists of two annuli (according to the multiplicities of the generic linear form) and each of them double-covers the disk image. Then the restriction of ℓ to $X_{5/4}$ has degree 4 while the total degree of the cover $\ell|_X$ is 6. Therefore either $X_{6/5}$ or X_1 overlaps $X_{5/4}$. Moreover, the restriction of ℓ to $X_{6/5}$ also has degree 4 according to the multiplicity 10 at the corresponding vertices. Therefore X_1 overlaps $X_{5/4}$.

We now show step by step the construction of the carrousel section.

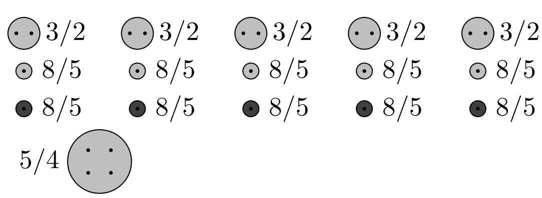
- $q_1 = 8/5$. $X_{8/5}$ consists of two polar zones. We represent them with two different gray colors in the carrousel section.



- $q_2 = 3/2$

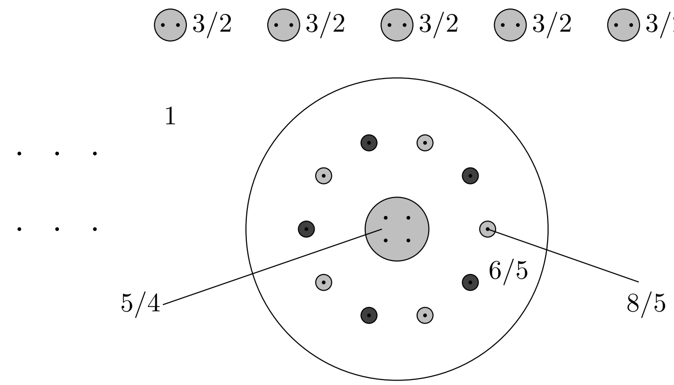


- $q_3 = 5/4$



- $q_4 = 6/5$ and $q_5 = 1$

We obtain the complete carrousel section at step $q_5 = 1$. The 6 points on the left represent 6 branching points in the zone X_1 and then 6 smooth transversal branches of the discriminant Δ. We read on this carrousel section that Δ has 14 branches and 12 distinct tangent lines $L_1, \ldots, L_{12}$ which decompose as follows:



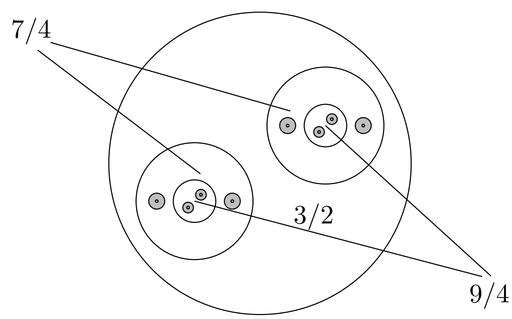
- three tangent branches $\delta, \delta', \delta''$ tangent to the same line L_1, each with single characteristic Puiseux exponent, respectively $6/5, 6/5$ and $5/4$, and Puiseux expansions:

  δ: $\quad u = av + bv^{6/5} + \dots$
  δ': $\quad u = av + b'v^{6/5} + \dots$
  δ'': $\quad u = av + b''v^{5/4} + \dots$

- For each $i = 2, \dots, 6$, one branch tangent to $L_i = \{u = a_i v\}$ with $3/2$ as single characteristic Puiseux exponent: $u = a_i v + b_i v^{3/2} + \dots$.

We now give an example of the phenomenon of nesting described in the proof of Proposition 9.1.

Example 9.4. Consider the hypersurface singularity in $(\mathbb{C}^3, 0)$ with equation $z^2 - f(x,y) = 0$ where $f(x,y) = 0$ is the equation of a plane curve having two branches δ_1 and δ_2 with Puiseux expansions respectively: $y = x^{3/2} + x^{7/4}$ and $y = x^{3/2} + x^{9/4}$ *i.e.*, the discriminant of $\ell = (x,y)$ is $\Delta = \delta_1 \cup \delta_2$. Then $X_{3/2}$ has two neighbour zones $X_{7/4}$ and $X_{9/4}$ and the inner boundary components of $\ell(X_{3/2})$ with rates $9/4$ are nested inside that of rate $7/4$. The carrousel section is given by the following picture.



Part 4: Zariski equisingularity implies semi-algebraic Lipschitz triviality

10. Equisingularity

In this section, we define Zariski equisingularity as in [27, Definition 1] and we specify it in codimension 2. We also define Lipschitz triviality in this codimension and we fix notations for the rest of the paper.

Definition 10.1. Let $(\mathfrak{X},0) \subset (\mathbb{C}^n,0)$ be a reduced hypersurface germ and $(Y,0) \subset (\mathfrak{X},0)$ a nonsingular germ. $\mathfrak{X}$ is *Zariski equisingular* along Y near 0 if for a generic projection $p\colon (\mathfrak{X},0) \to (\mathbb{C}^{n-1},0)$, the branch locus $\Delta \subset \mathbb{C}^{n-1}$ of p is equisingular along $p(Y)$ near 0 (in particular, $(p(Y),0) \subset (\Delta,0)$). When Y has codimension one in $\mathfrak{X}$, Zariski equisingularity means that Δ is nonsingular.

The notion of a generic projection is defined in [27, Section 3]. In [2], Briançon and Henry proved that when Y has codimension 2 in $\mathfrak{X}$, *i.e.*, for a family of complex hypersurfaces in $\mathbb{C}^3$, the definition remains unchanged if one considers for p the restriction of a generic linear projection $\mathscr{L}\colon \mathbb{C}^n \to \mathbb{C}^{n-1}$.

When Y has codimension one in $\mathfrak{X}$, Zariski equisingularity is equivalent to Whitney conditions for the pair $(\mathfrak{X} \smallsetminus Y, Y)$ and also to topological triviality of $\mathfrak{X}$ along Y.

From now on we consider the case where $(Y,0)$ is the singular locus of X and Y has codimension 2 in $\mathfrak{X}$ (*i.e.*, dimension $n-3$). Then any slice of $\mathfrak{X}$ by a smooth 3-space transversal to Y is a normal surface singularity. Summarizing the above, we have the following characterization of Zariski equisingularity for families of isolated singularities in $(\mathbb{C}^3,0)$:

Proposition 10.2. *Let $(\mathfrak{X},0) \subset (\mathbb{C}^n,0)$ be a reduced hypersurface germ at the origin of $\mathbb{C}^n$ with 2-codimension smooth singular locus $(Y,0)$. $\mathfrak{X}$ is Zariski equisingular along Y near 0 if for a generic linear projection $\mathscr{L}\colon \mathbb{C}^n \to \mathbb{C}^{n-1}$, the branch locus $\Delta \subset \mathbb{C}^{n-1}$ of $\mathscr{L}|_{\mathfrak{X}}\colon \mathfrak{X} \to \mathbb{C}^{n-1}$ is topologically equisingular along $\mathscr{L}(Y)$ near 0.* $\square$

Definition 10.3.
1. $(\mathfrak{X},0)$ *has constant (semi-algebraic) Lipschitz geometry* along Y if there exists a smooth (semi-algebraic) retraction $r\colon (\mathfrak{X},0) \to (Y,0)$ whose fibers are transverse to Y and a neighbourhood U of 0 in Y such that for all $y \in U$, there exists a (semi-algebraic) bilipschitz diffeomorphism $h_y\colon (r^{-1}(y),y) \to (r^{-1}(0) \cap \mathfrak{X},0)$.
2. The germ $(\mathfrak{X},0)$ is (semi-algebraic) *Lipschitz trivial* along Y if there exists a germ at 0 of a (semi-algebraic) bilipschitz homeomorphism $\Phi\colon (\mathfrak{X},Y) \to (X,0)\times Y)$ with $\Phi|_Y = id_Y$, where $(X,0)$ is a normal complex surface germ.

The aim of this last part is to prove the main result of the paper:

Theorem 10.4 (Theorem 1.1 in Introduction)**.** *The following are equivalent:*
1. $(\mathfrak{X},0)$ *is Zariski equisingular along Y;*
2. $(\mathfrak{X},0)$ *has constant semi-algebraic Lipschitz geometry along Y;*
3. $(\mathfrak{X},0)$ *is semi-algebraic Lipschitz trivial along Y*

The implication (3) $\Rightarrow$ (2) is trivial. (2) $\Rightarrow$ (1) is an easy consequence of part (2) of Theorem 1.3:

Proof of (2) $\Rightarrow$ (1). Assume $(\mathfrak{X},0)$ has constant semi-algebraic Lipschitz geometry along Y. Let $\mathscr{L}\colon \mathbb{C}^n \to \mathbb{C}^{n-1}$ be a general linear projection for $\mathfrak{X}$. Let $r\colon \mathscr{L}(\mathfrak{X}) \to$

$\mathscr{L}(Y)$ be a smooth semi-algebraic retraction whose fibers are transversal to $\mathscr{L}(Y)$. Its lift by $\mathscr{L}$ is a retraction $\widetilde{r}\colon \mathfrak{X} \to Y$ whose fibers are transversal to Y. For any $t \in Y$ sufficiently close to 0, $X_t = (\tilde{r})^{-1}(t)$ is semi-algebraically bilipschitz equivalent to $X_0 = (\tilde{r})^{-1}(0)$. Then, according to part (2) of Theorem 1.3, the discriminants Δ_t of the restrictions $\mathscr{L}|_{X_t}$ have same embedded topology. This proves that $\mathfrak{X}$ is Zariski is equisingular along Y. $\square$

The last sections of the paper are devoted to the proof of (1) $\Rightarrow$ (3).

Notations. Assume $\mathfrak{X}$ is Zariski equisingular along Y at 0. Since Zariski equisingularity is a stable property under analytic isomorphism $(\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$, we will assume without lost of generality that $Y = \{0\} \times \mathbb{C}^{n-3} \subset \mathbb{C}^3 \times \mathbb{C}^{n-3} = \mathbb{C}^n$ where 0 is the origin in $\mathbb{C}^3$. We will denote by $(\underline{\mathrm{x}}, t)$ the coordinates in $\mathbb{C}^3 \times \mathbb{C}^{n-3}$, with $\underline{\mathrm{x}} = (x, y, z) \in \mathbb{C}^3$ and $t \in \mathbb{C}^{n-3}$. For each t, we set $X_t = \mathfrak{X} \cap (\mathbb{C}^3 \times \{t\})$ and consider $\mathfrak{X}$ as the $n-3$-parameter family of isolated hypersurface singularities $(X_t, (0,t)) \subset (\mathbb{C}^3 \times \{t\}, (0,t))$. For simplicity, we will write $(X_t, 0)$ for $(X_t, (0,t))$.

Let $\mathscr{L}\colon \mathbb{C}^n \to \mathbb{C}^{n-1}$ be a generic linear projection for $\mathfrak{X}$. We choose the coordinates $(\underline{\mathrm{x}}, t)$ in $\mathbb{C}^3 \times \mathbb{C}^{n-3}$ so that $\mathscr{L}$ is given by $\mathscr{L}(x, y, z, t) = (x, y, t)$. For each t, the restriction $\mathscr{L}|_{\mathbb{C}^3 \times \{t\}}\colon \mathbb{C}^3 \times \{t\} \to \mathbb{C}^2 \times \{t\}$ is a generic linear projection for X_t. We denote by $\Pi_t \subset X_t$ the polar curve of the restriction $\mathscr{L}|_{X_t}\colon X_t \to \mathbb{C}^2 \times \{t\}$ and by Δ_t its discriminant curve $\Delta_t = \mathscr{L}(\Pi_t)$.

Throughout Part 4, we will use the Milnor balls defined as follows. By Speder [17], Zariski equisingularity implies Whitney conditions. Therefore one can choose a constant size $\epsilon > 0$ for Milnor balls of the $(X_t, 0)$ as t varies in some small ball D_δ about $0 \in \mathbb{C}^{n-3}$ and the same is true for the family of discriminants Δ_t. So we can actually use a "rectangular" Milnor ball as introduced in Section 5:

$$B^6_\epsilon = \{(x, y, z) : |x| \le \epsilon, |(y, z)| \le R\epsilon\},$$

and its projection

$$B^4_\epsilon = \{(x, y) : |x| \le \epsilon, |y| \le R\epsilon, \},$$

where R and ϵ_0 are chosen so that for $\epsilon \le \epsilon_0$ and t in a small ball D_δ,

(1) $B^6_\epsilon \times \{t\}$ is a Milnor ball for $(X_t, 0)$;
(2) for each α such that $|\alpha| \le \epsilon$, $h_t^{-1}(\alpha)$ intersects the standard sphere $\mathbb{S}^6_{R\epsilon}$ transversely, where h_t denotes the restriction $h_t = x|_{X_t}$;
(3) the curve Π_t and its tangent cone $T_0\Pi_t$ meet $\partial B^6_\epsilon \times \{t\}$ only in the part $|x| = \epsilon$.

11. Proof outline that Zariski implies bilipschitz equisingularity

We start by outlining proof that Zariski equisingularity implies semi-algebraic Lipschitz triviality. We assume therefore that we have a Zariski equisingular family $\mathfrak{X}$ as above.

We choose $B^6_\epsilon \subset \mathbb{C}^3$ and $D_\delta \subset \mathbb{C}^{n-3}$ as in the previous section. We want to construct a semi-algebraic bilipschitz homeomorphism $\Phi\colon \mathfrak{X} \cap (B^6_\epsilon \times D_\delta) \to (X_0 \times \mathbb{C}^{n-3}) \cap (B^6_\epsilon \times D_\delta)$ which preserves the t-parameter. Our homeomorphism will be a piecewise diffeomorphism.

We first prove the constancy of the polar zone rates as t varies.

In Section 5 we used a carrousel decomposition of B^4_ϵ adapted to the discriminant curve Δ_0 of the generic linear projection $\mathscr{L}|_{X_0}\colon X_0 \to B^4_\epsilon$ and we lifted it to obtain a decomposition of $(X, 0)$. Here, we will consider this construction for each

$\mathscr{L}|_{X_t}: X_t \to B^4_\epsilon \times \{t\}$. We also use a decomposition slightly different from that of Section 5 and Part 3: here we only perform the amalgamation of empty D-pieces and not that of the A-pieces (see Section 5 for definitions). We then consider a refined decomposition of X_t as a union of B- D- and A-pieces.

Next, we construct a semi-algebraic bilipschitz map $\phi\colon B^4_\epsilon \times D_\delta \to B^4_\epsilon \times D_\delta$ which restricts for each t to a map $\phi_t\colon B^4_\epsilon \times \{t\} \to B^4_\epsilon \times \{t\}$ which takes the carrousel decomposition of B^4_ϵ for X_t to the corresponding carrousel decomposition for X_0. We also arrange that ϕ_t preserves a foliation of B^4_ϵ adapted to the carrousels. A first approximation to the desired map $\Phi\colon \mathfrak{X} \cap (B^6_\epsilon \times D_\delta) \to (X_0 \times \mathbb{C}^{n-3}) \cap (B^6_\epsilon \times D_\delta)$ is then obtained by simply lifting the map ϕ via the branched covers $\mathscr{L}$ and $\mathscr{L}|_{X_0} \times id_{D_\delta}$:

$$\begin{array}{ccc} \mathfrak{X} \cap (B^6_\epsilon \times D_\delta) & \xrightarrow{\ \Phi\ } & (X_0 \cap B^6_\epsilon) \times D_\delta \\ {\scriptstyle \mathscr{L}}\Big\downarrow & & \Big\downarrow{\scriptstyle \mathscr{L}|_{X_0} \times id_{D_\delta}} \\ B^4_\epsilon \times D_\delta & \xrightarrow{\ \phi\ } & B^4_\epsilon \times D_\delta \end{array}$$

Now, fix t in D_δ. For $\underline{x} \in X_t \cap B^6_\epsilon$, denote by $K(\underline{x}, t)$ the local bilipschitz constant of the projection $\mathscr{L}|_{X_t}: X_t \to B^4_\epsilon \times \{t\}$ (see Section 4). Let us fix a large $K_0 > 0$ and consider the set $\mathcal{B}_{K_0,t} = \{\underline{x} \in X_t \cap (B^6_\epsilon \times D_\delta) \ :\ K(\underline{x}, t) \geq K_0\}$.

We then show that Φ is bilipschitz for the inner metric except possibly in the zone $C_{K_0} = \bigcup_{t \in D_\delta} \mathcal{B}_{K_0,t}$, where the bilipschitz constant for the linear projection $\mathscr{L}$ becomes large. This is all done in Section 13.

Using the approximation Lemma 4.7 of $\mathcal{B}_{K_0,t}$ by very thin zones, we then prove that Φ can be adjusted if necessary to be bilipschitz on C_{K_0}.

Once Φ is inner bilipschitz, we must show that the outer geometry is also preserved. This is based on the use of *test-curves*, which are defined in Section 14. The invariance of bilipschitz type of these curves when lifted to X_t enables one to test invariance of the outer geometry. This invariance is proved in Section 16 using a formula of Teissier [19] that we recall in section 15, and the pieces of the proof are finally put together in Section 17.

12. Constancy of the polar zone rates

In this section, we consider a very thin zone VT_t about a component Π'_t of Π_t, *i.e.*, a zone saturated by a family of polars $\Pi'_t(s)$ with s in a neighbourhood of $0 \in \mathbb{C}$ as the linear projection is changed (see definition 4.6 and also [1, Corollary 3.4] for the existence of the saturating family of polars). Put $\Delta'_t(s) = \mathscr{L}|_{X_t}(\Pi'_t(s))$, so $\Delta'_t = \Delta'_t(0)$.

Let $y(t,s) = \sum_i c_i(t,s)x^{p_i}$ be a Puiseux expansion for $\Delta'_t(s)$. Let $q = q(\Delta'_t)$ be the polar zone rate around Π'_t, *i.e.*, the exponent of the first term in the above Puiseux expansion of $\Delta'_t(s)$ which varies as s varies.

Lemma 12.1. *The polar zone rate $q(\Delta'_t)$ is constant for t sufficiently small.*

Proof. The exponent $q(\Delta'_t)$ has the same denominator as the last essential Puiseux exponent of $\Delta'_t(0)$ since $\Delta'_t(s)$ is a parallel curve to $\Delta'_t(0)$ (*i.e.*, the link $\Delta'_t(s) \cap \{x = \epsilon\}$ is a $(1,r)$-cable on $\Delta'_t(0) \cap \{x = \epsilon\}$ for some integer r). The numerator of $q(\Delta'_t)$, the cabling coefficient r and the linking number of $\Delta'_t(0) \cap \{x = \epsilon\}$ with $\Delta'_t(s) \cap \{x = \epsilon\}$ mutually determine each other. As t varies the linking number of $\Delta'_t(0) \cap \{x = \epsilon\}$ with $\Delta'_t(s) \cap \{x = \epsilon\}$ is constant, so the numerator of $q(\Delta'_t)$ is constant. $\square$

13. Foliated Carrousel

We use again the notations of Section 10. Our aim is to construct a bilipschitz homeomorphism of germs $\Phi\colon(\mathfrak{X},0)\to(X_0\times\mathbb{C}^{n-3},0)$. As described in Section 11, our first step is to construct a self-map $\phi\colon B^4_\epsilon\times D_\delta\to B^4_\epsilon\times D_\delta$ of the image of the generic linear projection which removes the dependence on t of the carrousels. So we first present a parametrized version of the carrousel construction of section 2.

Carrousel depending on the parameter t. By assumption, the family of discriminants Δ_t is an equisingular family of plane curves. It follows that the family of tangent spaces $T_0\Delta_t$ is also equisingular. For each t, $T_0\Delta_t$ is a union of tangent lines $L_t^{(1)},\dots,L_t^{(m)}$ and these are distinct lines for each t. Moreover the union $\Delta_t^{(j)}\subset\Delta_t$ of components of Δ_t which are tangent to $L_t^{(j)}$ is also equisingular.

In Section 2 we described the carrousel decomposition when there is no parameter t. It now decomposes a conical neighbourhood of each line $L_t^{(j)}$. These conical neighbourhoods are as follows. Let the equation of the j-th line be $y=a_1^{(j)}(t)x$. We choose a small enough $\eta>0$ and $\delta>0$ such that cones

$$V_t^{(j)}:=\{(x,y):|y-a_1^{(j)}(t)x|\le\eta|x|,|x|\le\epsilon\}\subset\mathbb{C}^2_t$$

are disjoint for all $t\in D_\delta$, and then shrink ϵ if necessary so $\Delta_t^{(j)}\cap\{|x|\le\epsilon\}$ will lie completely in $V_t^{(j)}$ for all t.

We now describe how the carrousel decomposition in section 2 is modified to the parametrized case. We fix $j=1$ for the moment and therefore drop the superscripts, so our tangent line L has equation $y=a_1(t)x$.

As before, we first truncate the Puiseux series for each component of Δ_t at a point where truncation does not affect the topology of Δ_t. Then for each pair $\kappa=(f,p_k)$ consisting of a Puiseux polynomial $f=\sum_{i=1}^{k-1}a_i(t)x^{p_i}$ and an exponent p_k for which there is a Puiseux series $y=\sum_{i=1}^{k}a_i(t)x^{p_i}+\dots$ describing some component of Δ_t, we consider all components of Δ_t which fit this data. If $a_{k1}(t),\dots,a_{km_\kappa}(t)$ are the coefficients of x^{p_k} which occur in these Puiseux polynomials we define

$$\begin{aligned}B_\kappa(t):=\Big\{(x,y):\alpha_\kappa|x^{p_k}|\le|y-\sum_{i=1}^{k-1}a_i(t)x^{p_i}|\le\beta_\kappa|x^{p_k}|\\ |y-(\sum_{i=1}^{k-1}a_i(t)x^{p_i}+a_{kj}(t)x^{p_k})|\ge\gamma_\kappa|x^{p_k}|\text{ for }j=1,\dots,m_\kappa\Big\}.\end{aligned}$$

Again, $\alpha_\kappa,\beta_\kappa,\gamma_\kappa$ are chosen so that $\alpha_\kappa<|a_{kj}(t)|-\gamma_\kappa<|a_{kj}(t)|+\gamma_\kappa<\beta_\kappa$ for each $j=1,\dots,m_\kappa$ and all small t. If ϵ is small enough, the sets $B_\kappa(t)$ will be disjoint for different κ. The closure of the complement in V_t of the union of the $B_\kappa(t)$'s is a union of A- and D-pieces.

Finally, by Lemma 12.1, we can refine the carrousel decomposition by adding additional D-pieces which are the projection by the maps $\mathscr{L}|_{X_t}$ of a continuous family of very thin zones about the polar curves Π_t as in Section 2. Lemma 12.1 guarantees that the rates of these D-pieces are constant through the family.

We then have constructed for each $t\in D_\delta$ a carrousel decomposition for V_t adapted to the discriminant curve $\Delta_t\cap V_t$ and projections of the very thin zones.

Foliated carrousel. We now refine our carrousel decomposition by adding a piecewise smooth foliation of V_t by complex curves compatible with the carrousel. We do this as follows:

A piece $B_\kappa(t)$ as above is foliated with closed leaves given by curves of the form

$$C_\alpha := \{(x,y) : y = \sum_{i=1}^{k-1} a_i(t)x^{p_i} + \alpha x^{p_k}\}$$

for $\alpha \in \mathbb{C}$ satisfying $\alpha_\kappa \le |\alpha| \le \beta_\kappa$ and $|a_{kj}(t) - \alpha| \ge \gamma_\kappa$ for $j = 1, \dots, m_\kappa$ and all small t. We foliate D-pieces similarly, including the pieces which correspond to very thin zones about Δ_t.

An A-piece has the form

$$A = \{(x,y) : \beta_1|x^{p_{k+1}}| \le |y - (\sum_{i=1}^{k} a_i(t)x^{p_i})| \le \beta_0|x^{p_k}|\}\,,$$

where $a_k(t)$ may be 0 and $p_{k+1} > p_k$ and β_0, β_1 are > 0. We foliate with leaves the immersed curves of the following form

$$C_{r,\theta} := \{(x,y) : y = \sum_{i=1}^{k} a_i(t)x^{p_i} + \beta(r)e^{i\theta}x^r\}$$

with $p_k \le r \le p_{k+1}$, $\theta \in \mathbb{R}$ and $\beta(r) = \beta_1 + \frac{p_{k+1}-r}{p_{k+1}-p_k}(\beta_0 - \beta_1)$. Note that these leaves may not be closed; for irrational r the topological closure is homeomorphic to the cone on a torus.

Definition 13.1. We call a carrousel decomposition equipped with such a foliation by curves a *foliated carrousel decomposition*.

Trivialization of the family of foliated carrousels. We set

$$V := \bigcup_{t \in D_\delta} V_t \times \{t\}$$

Proposition 13.2. *If δ is sufficiently small, there exists a semi-algebraic bilipschitz map $\phi_V : V \to V_0 \times D_\delta$ such that:*

1. *ϕ_V preserves the x and t-coordinates,*
2. *for each $t \in D_\delta$, ϕ_V preserves the foliated carrousel decomposition of V_t i.e., it maps the carrousel decomposition of V_t to that of V_0, preserving the respective foliations.*
3. *ϕ_V maps complex lines to complex lines on the portion $|x| < \epsilon$ of ∂V.*

Proof. We first construct the map on the slice $\{x = \epsilon\} \cap V$. Denote the carrousel section $V_t \cap \{x = \epsilon\}$ by $\mathcal{C}_t$. We start by extending the identity map $\mathcal{C}_0 \to \mathcal{C}_0$ to a family of piecewise smooth maps $\mathcal{C}_t \to \mathcal{C}_0$ which maps carrousel sections to carrousel sections.

For fixed $t = t_0$ we are looking at a carrousel section $\mathcal{C}_t$ as exemplified in Figure 1. The various regions are bounded by circles. Each disk or annulus in $\mathcal{C}_t$ is isometric to its counterpart in $\mathcal{C}_0$ so we map it by a translation. Each $B_\kappa(t) \cap \{x = \epsilon\}$ has a free action of the cyclic group $\mathbb{Z}/p_\kappa$ given by the first return map of its foliation as $\arg(x)$ varies, and we map each such region to the corresponding $B_\kappa(0) \cap \{x = \epsilon\}$ equivariantly with respect to its $\mathbb{Z}/p_\kappa$ action. This defines our map ϕ_V on $\{x = \epsilon\}$ and the requirement that ϕ_V preserve x-coordinate and foliation extends it uniquely

to all of $V \times D_\delta$ (note that the cross-sectional annuli and disks of A- and D-regions then always map by translations).

The pieces of the carrousel decomposition of V are semi-algebraic and ϕ_V maps their intersections with C_t to C_0 by translations. Therefore the map $C_t \to C_t$ is semi-algebraic as well as its equivariant extension to $\{x = \epsilon\}$ and the unique extension to the whole $V \times D_\delta$.

It remains to prove that the resulting map is bilipschitz. It is bilipschitz on $(V \cap \{|x| = \epsilon\}) \times D_\delta$ since it is piecewise smooth on a compact set. Depending what leaf of the foliation one is on, in a section $|x| = \epsilon'$ with $0 < \epsilon' \leq \epsilon$, a neighbourhood of a point $\underline{p}$ scales from a neighbourhood of a point $\underline{p}'$ with $|x| = \epsilon$ by a factor of $(\epsilon'/\epsilon)^r$ in the y-direction (and 1 in the x direction) as one moves in along a leaf, and the same for $\phi_V(\underline{p})$. So to high order the local bilipschitz constant of Φ_V at $\underline{p}$ is the same as for $\underline{p}'$ and hence bounded. Thus the bilipschitz constant is globally bounded on $(V \smallsetminus \{0\}) \times D_\delta$, and hence on $V \times D_\delta$. $\square$

The V of the above proposition was any one of the m sets $V^{(i)} := \bigcup_{t \in D_\delta} V_t^{(i)}$, so the proposition extends ϕ to all these sets. We extend the carrousel foliation on the union of these sets to all of $B^4_\epsilon \times \{t\}$ by foliating the complement of the union of $V_t^{(i)}$'s by complex lines. We denote B^4_ϵ with this carrousel decomposition and foliation structure by $B_{\epsilon,t}$. We finally extend ϕ to the whole of $\bigcup_{t \in D_\delta} B_{\epsilon,t}$ by a diffeomorphism which takes the complex lines of the foliation linearly to complex lines of the foliation on $B_{\epsilon,0} \times D_\delta$, preserving the x coordinate. The resulting map remains obviously semi-algebraic and bilipschitz.

We then have shown:

Proposition 13.3. *There exists a map $\phi\colon \bigcup_{t \in D_\delta} B_{\epsilon,t} \to B_{\epsilon,0} \times D_\delta$ which is semi-algebraic and bilipschitz and such that each $\phi_t\colon B_{\epsilon,t} \to B_{\epsilon,0} \times \{t\}$ preserves the carrousel decompositions and foliations.* $\square$

Corollary 13.4. *There exists a commutative diagram*

$$\begin{array}{ccc}
(\mathfrak{X}, 0) \cap (\mathbb{C}^3 \times D_\delta) & \xrightarrow{\Phi} & (X_0, 0) \times D_\delta \\
{\scriptstyle \mathscr{L}|}\big\downarrow & & \big\downarrow{\scriptstyle \mathscr{L}|_{X_0} \times id} \\
(\mathbb{C}^2, 0) \times D_\delta & \xrightarrow{\phi} & (\mathbb{C}^2, 0) \times D_\delta
\end{array}$$

such that ϕ is the map of the above proposition and Φ is inner bilipschitz except possibly in the set C_{K_0} and semi-algebraic.

Proof. Φ is simply the lift of ϕ over the branched cover p. The map p is a local diffeomorphism with Lipschitz constant bounded above by K_0 outside C_{K_0} so Φ has Lipschitz coefficient bounded by K_0^2 times the Lipschitz bound for ϕ outside C_{K_0}. The semi-algebraicity of Φ is because ϕ, p and $\mathscr{L}|_{X_0} \times id$ are semi-algebraic. $\square$

14. Test curves

Definition 14.1. Let $(X, 0) \subset (\mathbb{C}^3, 0)$ be an isolated hypersurface singularity and let $\ell\colon \mathbb{C}^3 \to \mathbb{C}^2$ be a generic linear projection for $(X, 0)$. A *test curve* is a complex curve $(\gamma, 0) \subset (\mathbb{C}^2, 0)$ which is a leaf of the foliated carrousel decomposition adapted to the discriminant curve of $\ell|_X$. A *lifted test curve* is a curve $L_\gamma := (\ell|_X)^{-1}(\gamma)$ where $(\gamma, 0) \subset (\mathbb{C}^2, 0)$ is a test curve.

Definition 14.2. If $\ell' \colon \mathbb{C}^3 \to \mathbb{C}^2$ is a linear projection which is generic for L_γ we call the plane curve $P_\gamma = \ell'(L_\gamma)$ a *projection* of the lifted test curve L_γ.

Notice that the topological type of $(P_\gamma, 0)$ does not depend on the choice of ℓ'.

Let us choose the coordinates (x, y, z) of $\mathbb{C}^3$ in such a way that $\ell = (x, y)$ and γ is tangent to the x–axis. We consider a Puiseux expansion of γ:

$$\begin{aligned} x(w) &= w^q \\ y(w) &= a_1 w^{q_1} + a_2 w^{q_2} + \ldots + a_{n-1} w^{q_{n-1}} + a_n w^{q_n} + \ldots \end{aligned}$$

Let $F(x, y, z) = 0$ be an equation for $X \subset \mathbb{C}^3$.

Definition 14.3. We call the plane curve $(F_\gamma, 0) \subset (\mathbb{C}^2, 0)$ with equation

$$F(x(w), y(w), z) = 0$$

a *spreading* of the lifted test curve L_γ.

Notice that the topological type of $(F_\gamma, 0)$ does not depend on the choice of the parametrization.

Lemma 14.4. *Let γ be a test curve of $(X, 0)$. The topological types of the spreading $(F_\gamma, 0)$ and of the projection $(P_\gamma, 0)$ determine each other.*

Proof. Assume that the coordinates of $\mathbb{C}^3$ are chosen in such a way that $\ell' = (x, z)$. Then $P_\gamma = \rho(F_\gamma)$ where $\rho \colon \mathbb{C}^2 \to \mathbb{C}^2$ denotes the morphism $\rho(w, z) = (w^q, z)$, which is a cyclic q-fold cover branched on the line $w = 0$. Thus P_γ determines F_γ. The link of $P_\gamma \cup \{z\text{-axis}\}$ consists of an iterated torus link braided around an axis and it has a $\mathbb{Z}/q$ action fixing the axis. Such a $\mathbb{Z}/q$ action is unique up to isotopy. Thus the link of F_γ can be recovered from the link of P_γ by quotienting by this $\mathbb{Z}/q$–action. $\square$

15. Restriction formula and spreadings

In this section, we apply a restriction formula to compute the Milnor number of a spreading of a lifted test curve. Let us first recall the formula.

Proposition 15.1. ([19, 1.2]) *Let $(Y, 0) \subset (\mathbb{C}^{n+1}, 0)$ be a germ of hypersurface with isolated singularity, let H be a hyperplane of $(\mathbb{C}^{n+1}, 0)$ such that $H \cap Y$ has isolated singularity. Assume that $(z_0, \ldots, z_n)$ is a system of coordinates of $\mathbb{C}^{n+1}$ such that H is given by $z_0 = 0$. Let* $\mathrm{proj} \colon (Y, 0) \to (\mathbb{C}, 0)$ *be the restriction of the function z_0. Then*

$$m_H = \mu(Y) + \mu(Y \cap H),$$

where $\mu(Y)$ and $\mu(Y \cap H)$ denote the Milnor numbers respectively of $(Y, 0) \subset (\mathbb{C}^{n+1}, 0)$ and $(Y \cap H, 0) \subset (H, 0)$, and where m_H is the multiplicity of the origin 0 as the discriminant of the morphism proj.

The multiplicity m_H is defined as follow ([19]). Assume that an equation of $(Y, 0)$ is $f(z_1, \ldots, z_n) = 0$, and let $(\Gamma, 0) \subset (Y, 0)$ be the curve in $(\mathbb{C}^{n+1}, 0)$ defined by $\frac{\partial f}{\partial z_1} = \ldots = \frac{\partial f}{\partial z_n} = 0$. Then $m_H = (Y, \Gamma)_0$, the intersection at the origin of the two germs $(Y, 0)$ and $(\Gamma, 0)$.

Remark 15.2. Applying this when $n = 1$, *i.e.*, in the case of a plane curve $(Y, 0) \subset (\mathbb{C}^2, 0)$, we obtain:

$$m_H = \mu(Y) + \mathrm{mult}(Y) - 1$$

where $\mathrm{mult}(Y)$ denotes the multiplicity of the plane curve germ $(Y, 0)$.

Proposition 15.3. *Let $(X,0) \subset (\mathbb{C}^3,0)$ be an isolated hypersurface singularity. Let $\ell\colon \mathbb{C}^3 \to \mathbb{C}^2$ be a generic linear projection for $(X,0)$ and $(\gamma,0) \subset (\mathbb{C}^2,0)$ a test curve. The Milnor number at 0 of the spreading $(F_\gamma,0)$ can be computed in terms of the following data:*

1. *the multiplicity of the surface $(X,0)$;*
2. *the topological type of the triple (X,Π,L_γ) where $(\Pi,0) \subset (X,0)$ denotes the polar curve of ℓ.*

Proof. Choose coordinates in $\mathbb{C}^3$ such that $\ell = (x,y)$, and consider a Puiseux expansion of γ as in Section 14. Applying the restriction formula 15.1 to the projection $\mathrm{proj}\colon (F_\gamma,0) \to (\mathbb{C},0)$ defined as the restriction of the function w, we obtain

$$\mu(F_\gamma) = (F_\gamma,\Gamma)_0 - \mathrm{mult}(F_\gamma) + 1$$

where $(\Gamma,0)$ has equation

$$\frac{\partial F}{\partial z}(x(w),y(w),z) = 0$$

We have easily: $\mathrm{mult}(F_\gamma) = \mathrm{mult}(X,0)$.

Moreover, the polar curve Π of the projection $\ell|_X$ is the set $\{(x,y,z) \in (X,0) : \frac{\partial F}{\partial z}(x,y,z) = 0\}$. Since the map $\theta\colon F_\gamma \to L_\gamma$ defined by $\theta(w,z) = (x(w),y(w),z)$ is a bijection, the intersection multiplicity $(F_\gamma,\Gamma)_0$ in $\mathbb{C}^2$ equals the intersection multiplicity at 0 on the surface $(X,0)$ of the two curves L_γ and Π. $\square$

16. Constancy of lifted test curves

We use again the notations of Section 10. The aim of this section is to prove the constancy through the family $(X_t,0)$ of the geometry of lifted test curves.

Proposition 16.1. *The following data is constant through the family $(X_t,0)$:*

1. *the multiplicity* $\mathrm{mult}(X_t,0)$ *of the surface $(X_t,0)$,*
2. *the topological type of the pair (X_t,Π_t).*

Proof. (1) follows (in all codimension) from the chain of implications: Zariski equisingularity $\Rightarrow$ Whitney conditions (Speder [17]) $\Rightarrow$ μ^*-constant (e.g., [6]; see also [23] (in codimension 2)). (2) follows immediately since (X_t,Π_t) is a continuous family of branched covers of $\mathbb{C}^2$ of constant covering degree and with constant topology of the branch locus. $\square$

Definition 16.2. Let $(X_t,0)$ be a Zariski equisingular family as before. If γ_0 is a test curve for X_0 then the family of curves $\gamma_t = \phi_t^{-1}(\gamma_0)$ (notation of Proposition 13.3) is called a *test curve family*.

Corollary 16.3. *Consider a test curve family (γ_t) and let $\mathscr{L}'\colon \mathbb{C}^3 \times \mathbb{C}^{n-3} \to \mathbb{C}^2 \times \mathbb{C}^{n-3}$ be a general linear projection for $(\mathfrak{X},0)$ which is also general for the family of lifted test curves $(L_{\gamma_t})_t$. Then the family of projections $P_{\gamma_t} = \mathscr{L}'(L_{\gamma_t})$ has constant topological type.* $\square$

Proof. According to Propositions 15.3 and 16.1, the Milnor number $\mu(F_{\gamma_t})$ is constant. The result follows by the Lê-Ramanujan theorem for a family of plane curves ([12]) and Lemma 14.4. $\square$

17. Proof that Zariski equisingularity implies Lipschitz triviality

We use the notations of Sections 10 and 11. For t fixed in D_δ and $K_0 > 0$ sufficiently large, recall (section 11) that $\mathcal{B}_{K_0,t}$ denote the neighbourhood of Π_t in X_t where the local bilipschitz constant $K(\underline{x}, t)$ of $\mathscr{L}|_{X_t}$ is bigger than K_0.

To complete the proof that Zariski equisingularity implies Lipschitz triviality we will show that Φ as in Corollary 13.4 is bilipschitz with respect to the outer metric after modifying it as necessary within the zone

$$C_{K_0} := \bigcup_{t\in D_\delta} \mathcal{B}_{K_0,t}$$

Proof. We first make the modification in the very thin zone $\mathcal{B}_{K_0,0}$. Recall that according to Lemma 4.7, $\mathcal{B}_{K_0,t}$ can be approximated for each t by a very thin zone about Π_t, and then its image $\mathcal{N}_{K_0,t} = \mathscr{L}|_{X_t}(\mathcal{B}_{K_0,t})$ can be approximated by a very thin zone about the polar curve Δ_t, *i.e.*, by a union of D-pieces of the foliated carrousel $B_{\epsilon,t}$.

Let $\mathcal{N}'_{K_0,0}$ be one of these D-pieces, and let q be its rate. By [1, Lemma 12.1.3], any component $\mathcal{B}'_{K_0,0}$ of $\mathcal{B}_{K_0,0}$ over $\mathcal{N}'_{K_0,0}$ is also a D(q)-piece in X_0.

By Proposition 13.3, the inverse image under the map ϕ_t of $\mathcal{N}'_{K_0,0}$ is $\mathcal{N}'_{K_0,t}$. Then the inverse image under the lifting Φ_t of $\mathcal{B}'_{K_0,0} \subset X_0$ will be the very thin zone $\mathcal{B}'_{K_0,t}$ in X_t with the same rate q. So we can adjust Φ_t as necessary in this zone to be a bilipschitz equivalence for the inner metric and to remain semi-algebraic.

We can thus assume now that Φ is semi-algebraic and bilipschitz for the inner metric, with Lipschitz bound K say. We will now show it is also bilipschitz for the outer metric.

We first consider a pair of points $\underline{p}_1, \underline{p}_2$ of $X_t \cap B^6_\epsilon$ which lie over the same point $\underline{p} \in B^4_\epsilon$ (we will say they are "vertically aligned"). We may assume, by moving them slightly if necessary, that $\underline{p}$ is on a closed curve γ of our foliation of B^4_ϵ with Puiseux expansion

$$\begin{aligned} x(w) &= w^q \\ y(w) &= a_1 w^{q_1} + a_2 w^{q_2} + \ldots + a_{n-1} w^{q_{n-1}} + a_n w^{q_n} + \ldots \end{aligned}$$

Let $\mathscr{L}'$ be a different generic linear projection as in Corollary 16.3. If $w = w_0$ is the parameter of the point $\underline{p} \in \gamma$, consider the arc $C(s) = (x(sw_0), y(sw_0))$, $s \in [0,1]$ and lift it to X_t to obtain a pair of arcs $\underline{p}_1(s), \underline{p}_2(s)$, $s \in [0,1]$ with $\underline{p}_1(1) = \underline{p}_1$ and $\underline{p}_2(1) = \underline{p}_2$. We assume that the distance $d(\underline{p}_1(s), \underline{p}_2(s))$ shrinks faster than linearly as $s \to 0$, since otherwise the pair is uninteresting from the point of view of bilipschitz geometry.

Note that the distance $d(\underline{p}_1(s), \underline{p}_2(s))$ is a multiple $kd(\mathscr{L}'(\underline{p}_1(s)), \mathscr{L}'(\underline{p}_2(s)))$ (with k depending only on the projections $\mathscr{L}$ and $\mathscr{L}'$). Now $d(\mathscr{L}'(\underline{p}_1(s)), \mathscr{L}'(\underline{p}_2(s)))$ is to high order $s^r d(\mathscr{L}'(\underline{p}_1(1)), \mathscr{L}'(\underline{p}_2(1)))$ for some rational $r > 1$. Moreover, by Corollary 16.3, if we consider the corresponding picture in X_0, starting with $\Phi(\underline{p}_1)$ and $\Phi(\underline{p}_2)$ we get the same situation with the same exponent r. So to high order we see that $d(\Phi(\underline{p}_1(s)), \Phi(\underline{p}_2(s)))/d(\underline{p}_1(s), \underline{p}_2(s))$ is constant.

There is certainly an overall bound on the factor by which Φ scales distance between vertically aligned points $\underline{p}_1$ and $\underline{p}_2$ so long as we restrict ourselves to the compact complement in $\mathfrak{X} \cap (B^6_\epsilon \times D_\delta)$ of an open neighbourhood of $\{0\} \times D_\delta$. The above argument then shows that this bound continues to hold as we move towards

0. Thus Φ distorts distance by at most a constant factor for all vertically aligned pairs of points.

Finally, consider any two points $\underline{p}_1, \underline{p}_2 \in X_0 \cap B^6_\epsilon$ and their images $\underline{q}_1, \underline{q}_2$ in $\mathbb{C}^2$. We will assume for the moment that neither $\underline{p}_1$ or $\underline{p}_2$ is in a very thin zone. The outer distance between $\underline{p}_1$ and $\underline{p}_2$ is the length of the straight segment joining them. Let γ be the image of this segment, so γ connects $\underline{q}_1$ to $\underline{q}_2$. If γ intersects the very thin zone, we can modify it to a curve avoiding the very thin zone which has length less than π times the original length of γ. Lift γ to a curve γ' starting at $\underline{p}_1$. Then γ' ends at a point $\underline{p}'_2$ which is vertically aligned with $\underline{p}_2$. Let γ'' be the curve obtained by appending the vertical segment $\underline{p}'_2\underline{p}_2$ to γ'. We then have:

$$L(\underline{p}_1\underline{p}_2) \leq L(\gamma''),$$

where L denotes length. On the other hand, $L(\gamma') \leq K_0 L(\gamma)$, and as γ is the projection of the segment $\underline{p}_1\underline{p}_2$, we have $L(\gamma) \leq L(\underline{p}_1\underline{p}_2)$, so $L(\gamma') \leq K_0 L(\underline{p}_1\underline{p}_2)$. If we join the segment $\underline{p}_1\underline{p}_2$ to γ' at $\underline{p}_1$ we get a curve from $\underline{p}_2$ to $\underline{p}'_2$, so $L(\underline{p}'_2\underline{p}_2) \leq (1+\pi K_0)L(\underline{p}_1\underline{p}_2)$ and as $L(\gamma'') = L(\gamma') + L(\underline{p}'_2\underline{p}_2)$, we then obtain :

$$L(\gamma'') \leq (1 + 2K_0\pi)L(\underline{p}_1\underline{p}_2)$$

We have thus shown that up to a bounded constant the outer distance between two points can be achieved by following a path in X followed by a vertical segment.

We have proved this under the assumption that we do not start or end in the very thin zone. Now, take two other projections $\mathscr{L}'$ and $\mathscr{L}''$ such that for K_0 is sufficiently large, the very thin zones of the restrictions of $\mathscr{L}$, $\mathscr{L}'$ and $\mathscr{L}''$ to X_0 with constant K_0 are pairwise disjoint outside the origin. Then if $\underline{p}_1$ and $\underline{p}_2$ are any two points in $X_0 \cap B^6_\epsilon$, they are outside the very thin zone for at least one of $\mathscr{L}$, $\mathscr{L}'$ or $\mathscr{L}''$ and we conclude as before.

The same argument applies to $X_0 \times D_\delta$. Now paths in $\mathfrak{X}$ are modified by at most a bounded factor by Φ since Φ is inner bilipschitz, while vertical segments are also modified by at most a bounded factor by the previous argument. Thus outer metric is modified by at most a bounded factor.

$\square$

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